THE FOURIER TRANSFORM ON HARMONIC MANIFOLDS OF PURELY EXPONENTIAL VOLUME GROWTH

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Abstract. Let X be a complete, simply connected harmonic manifold of purely exponential volume growth. This class contains all non-flat harmonic manifolds of non-positive curvature and, in particular all known examples of non-compact harmonic manifolds except for the flat spaces.

Denote by $h > 0$ the mean curvature of horospheres in X, and set $\rho = h/2$. Fixing a basepoint $o \in X$, for $\xi \in \partial X$, denote by B_ξ the Busemann function at ξ such that $B_\xi(o) = 0$. then for $\lambda \in \mathbb{C}$ the function $e^{(i\lambda - \rho)B_\xi}$ is an eigenfunction of the Laplace-Beltrami operator with eigenvalue $-(\lambda^2 + \rho^2)$.

For a function f on X, we define the Fourier transform of f by

$$\tilde{f}(\lambda, \xi) := \int_X f(x) e^{(-i\lambda - \rho)B_\xi(x)} dvol(x)$$

for all $\lambda \in \mathbb{C}, \xi \in \partial X$ for which the integral converges. We prove a Fourier inversion formula

$$f(x) = C_0 \int_0^\infty \int_{\partial X} \tilde{f}(\lambda, \xi) e^{(i\lambda - \rho)B_\xi(x)} d\lambda_o(\xi) |c(\lambda)|^{-2} d\lambda$$

for $f \in C_c^\infty(X)$, where c is a certain function on $\mathbb{R} - \{0\}$, λ_o is the visibility measure on ∂X with respect to the basepoint $o \in X$ and $C_0 > 0$ is a constant. We also prove a Plancherel theorem, and a version of the Kunze-Stein phenomenon.

Contents

1. Introduction

Throughout this article, all Riemannian manifolds are assumed to be complete and simply connected. A *harmonic manifold* is a Riemannian manifold X such that for any point $x \in X$, there exists a non-constant harmonic function on a punctured neighbourhood of x which is radial around x, i.e. only depends on the geodesic distance from x. Copson and Ruse showed that this is equivalent to requiring that sufficiently small geodesic spheres centered at x have constant mean curvature, and moreover such manifolds are Einstein manifolds [CR40]. Hence they have constant curvature in dimensions 2 and 3. The Euclidean spaces and rank one symmetric spaces are examples of harmonic manifolds. The Lichnerowicz conjecture asserts that conversely any harmonic manifold is either flat or locally symmetric of rank one. The conjecture was proved for harmonic manifolds of dimension 4 by A. G. Walker [Wal48]. In 1990 Z. I. Szabo proved the conjecture for compact simply connected harmonic manifolds [Sza90]. In 1995 G. Besson, G. Courtois and S. Gallot proved the conjecture for manifolds of negative curvature admitting a compact quotient [BCG95], using rigidity results from hyperbolic dynamics including the work of Y. Benoist, P. Foulon and F. Labourie [BFL92] and that of P. Foulon and F. Labourie [FL92]. Using their results it was shown in 2012 by the second author [Kni12] that the Lichnerowicz conjecture even holds for manifolds of non-positive curvature (or more generally no focal points) provided they admit a compact quotient. Furthermore, he also verified the conjecture for harmonic manifolds admitting a compact quotient with Gromov hyperbolic fundamental group.

In 2005 Y. Nikolayevsky proved the conjecture for harmonic manifolds of dimension 5, showing that these must in fact have constant curvature [Nik05]. Another fundamental result states that harmonic manifolds of subexponential volume growth are flat [RS02].

In 1992 however E. Damek and F. Ricci had already provided in the non-compact case a family of counterexamples to the Lichnerowicz conjecture, which have come to be known as *harmonic NA groups*, or *Damek-Ricci spaces* [DR92]. These are solvable Lie groups $X = NA$ with a suitable left-invariant Riemannian metric, given by the semi-direct product of a nilpotent Lie group N of *Heisenberg type* (see [Kap80]) with $A = \mathbb{R}^+$ acting on N by anisotropic dilations. While the non-compact rank one symmetric spaces G/K may be identified with harmonic NA groups (apart from the real hyperbolic spaces), there are examples of harmonic NA groups which are not symmetric. In 2006, J. Heber proved that the only complete simply connected homogeneous harmonic manifolds are the Euclidean spaces, rank one symmetric spaces, and harmonic NA groups [Heb06].

Though the harmonic NA groups are not symmetric in general, there is still a well developed theory of harmonic analysis on these spaces which parallels that of the symmetric spaces G/K. For a non-compact symmetric space $X = G/K$, an important role in the analysis on these spaces is played by the well-known *Helgason Fourier transform* [Hel94]. For harmonic NA groups, F. Astengo, R. Camporesi and B. Di Blasio have defined a Fourier transform [ACB97], which reduces to the Helgason Fourier transform when the space is symmetric. In both cases a Fourier inversion formula and a Plancherel theorem hold.

The aim of the present article is to generalize these results to a large class of non-compact harmonic manifolds. Our analysis will be concerned with harmonic manifolds of purely exponential volume growth which include all non-flat harmonic manifolds of non-positive sectional curvature or, more generally, all non-flat harmonic manifolds without focal points as was shown by the second author (see [Kni12, Theorem 6.5]). In particular, this class includes all known examples of non-flat and non-compact harmonic manifolds. By *purely exponential volume growth*, we mean that there are constants $C > 1$, $h > 0$ such that for all $R > 1$ the volume of metric balls $B(x, R)$ of radius R and center $x \in X$ is given by

$$\frac{1}{C}e^{hR} \leq vol(B(x,R)) \leq Ce^{hR}. \tag{1}$$

Let X be a simply connected noncompact harmonic manifold of purely exponential volume growth with a fixed basepoint $o \in X$. It was shown in [Kni12] that for such harmonic manifolds the condition of purely exponential volume growth is equivalent to either of the following three conditions:

1) X is Gromov hyperbolic.
2) X has rank one.
3) The geodesic flow of X is Anosov with respect to the Sasaki metric.

Moreover, it follows from the work in [KP16] that the Gromov boundary agrees with the visibility boundary ∂X introduced in [EO73]. The set $X \cup \partial X$ equipped with the cone topology defines a topological space homeomorphic to a closed unit ball in $\mathbb{R}^n$, where $n = \dim X$. For a given $\xi \in \partial X$ and any geodesic ray $\gamma : [0, \infty) \to X$ representing ξ (see section 2 for a precise definition) the Busemann function B_ξ with $B_\xi(o) = 0$ is given by

$$B_\xi(y) = \lim_{t\to\infty} (d(y, \gamma(t)) - d(o, \gamma(t))).$$

The level sets of B_ξ are called *horospheres* in X. The manifold X, being harmonic, is also *asymptotically harmonic*, i.e. the mean curvature of all horospheres is equal to a constant $h \geq 0$. If X has purely exponential volume growth then h is positive and agrees with the constant h appearing in (1). An easy computation shows that for $\rho = h/2$ and any $\lambda \in \mathbb{C}$ and $\xi \in \partial X$, the function $f = e^{(i\lambda-\rho)B_\xi}$ is an eigenfunction of the Laplace-Beltrami operator Δ on X with eigenvalue $-(\lambda^2 + \rho^2)$.

The Fourier transform of a function $f \in C_c^\infty(X)$ is then defined to be the function on $\mathbb{C} \times \partial X$ given by

$$\tilde{f}(\lambda, \xi) = \int_X f(x)e^{(-i\lambda-\rho)B_\xi(x)} dvol(x).$$

When X is a non-compact rank one symmetric space, this reduces to the Helgason Fourier transform.

The normalized canonical measure of the unit tangent sphere T_o^1X induced by the Riemannian metric is denoted by θ_o. The unit tangent sphere T_o^1X is identified with the boundary ∂X via the homeomorphism $pr_o : v \in T_o^1X \mapsto \xi = \gamma_v(\infty) \in \partial X$, where γ_v is the unique geodesic ray with $\gamma_v'(0) = v$. Pushing forward the measure θ_o on T_o^1X by the map pr_o gives a measure on ∂X called the *visibility measure*, which we denote by λ_o. We have the following Fourier inversion formula:

Theorem 1.1. *Let (X,g) be a simply connected, harmonic manifold of purely exponential volume growth. Then there is a constant $C_0 > 0$ and a function c on $\mathbb{C} - \{0\}$ such that for any $f \in C_c^\infty(X)$, we have*

$$f(x) = C_0 \int_0^\infty \int_{\partial X} \tilde{f}(\lambda,\xi) e^{(i\lambda-\rho)B_\xi(x)} d\lambda_o(\xi) |c(\lambda)|^{-2} d\lambda$$

for all $x \in X$.

We also obtain a Plancherel formula:

Theorem 1.2. *Let (X,g) be a simply connected, harmonic manifold of purely exponential volume growth. For any $f,g \in C_c^\infty(X)$, we have*

$$\int_X f(x)\overline{g(x)} dvol(x) = C_0 \int_0^\infty \int_{\partial X} \tilde{f}(\lambda,\xi)\overline{\tilde{g}(\lambda,\xi)} d\lambda_o(\xi)|c(\lambda)|^{-2} d\lambda.$$

The Fourier transform extends to an isometry of $L^2(X, dvol)$ into $L^2((0,\infty) \times \partial X, C_0 d\lambda_o(\xi)|c(\lambda)|^{-2} d\lambda)$.

The function c in the previous two theorems is holomorphic on $\operatorname{Im}\lambda < 0$ and has the following integral representation:

Theorem 1.3. *Let (X,g) be a simply connected harmonic manifold of purely exponential volume growth and c be the c-function of the radial hypergroup of X. Let $\operatorname{Im}\lambda < 0$. Then we have*

$$c(\lambda) = \int_{\partial X} e^{-2(i\lambda-\rho)(\xi|\eta)_x} d\lambda_x(\eta).$$

for any $x \in X, \xi \in \partial X$, where $(\xi|\eta)_x$ is the Gromov product on X given in Definition 2.3 below.

We define a notion of convolution with radial functions and prove the following version of the Kunze-Stein phenomenon:

Theorem 1.4. *Let (X,g) be a simply connected harmonic manifold of purely exponential volume growth. Let $x \in X$ and let $1 \le p < 2$. Let $g \in C_c^\infty(X)$ be radial around the point x. Then for any $f \in C_c^\infty(X)$ the inequality*

$$||f * g||_2 \le C_p ||g||_p ||f||_2$$

holds for some constant $C_p > 0$. It follows that for any $g \in L^p(X)$ radial around x, the map $f \in C_c^\infty(X) \mapsto f * g$ extends to a bounded linear operator on $L^2(X)$ with operator norm at most $C_p||g||_p$.

The article is organized as follows. In section 2 we recall basic facts about Gromov hyperbolic spaces and harmonic manifolds which we require. In section 3 we compute the action of the Laplacian Δ on spaces of functions constant on geodesic spheres and horospheres respectively. In section 4 we carry out the harmonic analysis of radial functions, i.e. functions constant on geodesic spheres centered around a given point. Unlike the well-known *Jacobi analysis* [Koo84] which applies to radial functions on rank one symmetric spaces and harmonic NA groups, our analysis here is based on *hypergroups* [BH95]. We define a spherical Fourier transform for radial functions, and obtain an inversion formula and Plancherel theorem for this transform. In section 5 we prove the inversion formula and Plancherel formula for the Fourier transform. The main point of the proof is an identity expressing radial eigenfunctions in terms of an integral over the boundary ∂X. The integral formula for the function c (Theorem 1.3) is proved in section 6. In section 7 we define an operation of convolution with radial functions, and show that the L^1 radial functions form a commutative Banach algebra under convolution. Finally in section 8 we prove a version of the Kunze-Stein phenomenon.

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2. Basics about harmonic manifolds and Gromov hyperbolic spaces

2.1. **Gromov hyperbolic spaces.** We recall briefly basic facts and definitions about Gromov hyperbolic spaces. Standard references for this section include [BH99], [BS07].

Let (X, d) be a metric space. We assume that X is *geodesic*, i.e. any two points x, y can be joined by a *geodesic*, which is an isometric embedding $\gamma : [0, T] \to X$ satisfying $\gamma(0) = x, \gamma(T) = y$, where $T = d(x, y)$. A geodesic metric space is called *Gromov hyperbolic* if there exists a $\delta > 0$ such that geodesic triangles are δ-thin, that is each side is contained in the union of the δ-neighbourhoods of the other two sides.

A *geodesic ray* is an isometric embedding $\gamma : [0, \infty) \to X$. Two geodesic rays γ_1, γ_2 are said to be equivalent if $\{d(\gamma_1(t), \gamma_2(t)) : t \geq 0\}$ is bounded. The *boundary at infinity* ∂X of X is defined to be the set of equivalence classes of geodesic rays in X. The equivalence class of a geodesic ray γ is denoted by $\gamma(\infty) \in \partial X$. For any $x \in X$ and $\xi \in \partial X$ there exists a geodesic ray γ with $\gamma(0) = x, \gamma(\infty) = \xi$. For any two distinct points $\xi, \eta \in \partial X$, there is a bi-infinite geodesic $\gamma : \mathbb{R} \to X$ such that $\gamma(-\infty) = \xi, \gamma(\infty) = \eta$. We remark that while these geodesics need not be unique, any two such geodesics lie within bounded distance of each other.

The *cone topology* on $\overline{X} := X \cup \partial X$ is defined as follows:

We fix an origin $o \in X$. A basis of neighbourhoods for a boundary point $\xi \in \partial X$ is given by sets of the form $U(\gamma, R, \epsilon)$, where γ is a geodesic ray with $\gamma(0) = o, \gamma(\infty) = \xi$, and $R, \epsilon > 0$, and where $U(\gamma, R, \epsilon)$ is the set of $y \in \overline{X} - \overline{B(o, R)}$ such that any geodesic α joining o to y satisfies $d(\alpha(R), \gamma(R)) < \epsilon$. Neighbourhoods of points $x \in X$ are the usual neighbourhoods for the metric topology on X.

The cone topology is independent of the choice of origin o. The space $\overline{X}$ is compact if and only if the metric space X is proper, that is closed and bounded balls in X are compact.

Given $x, y, z \in X$, the *Gromov product* of y, z with respect to x is defined by

$$(y|z)_x := \frac{1}{2}\left(d(x, y) + d(x, z) - d(y, z)\right).$$

For $\xi, \eta \in \overline{X}$ the Gromov product of ξ, η with respect to $x \in X$ is defined by

$$(\xi|\eta)_x := \liminf_{y \to \xi, z \to \eta} (y|z)_x \ \in [0, +\infty]$$

(where y, z tend to ξ, η in the cone topology). This extends the Gromov product to a function $(.|.)_x : \overline{X} \times \overline{X} \to [0, +\infty]$ which satisfies $(\xi|\eta)_x = +\infty$ if and only if $\xi = \eta \in \partial X$. Moreover, for a sequence $x_n \in \overline{X}$ and $\xi \in \partial X$, we have $x_n \to \xi$ (in the cone topology) if and only if $(x_n|\xi)_x \to +\infty$.

2.2. **Harmonic manifolds.** We present some fundamental facts about non-compact simply connected harmonic manifolds. References for this class of manifolds include [RWW61], [Sza90], [Wil93], [KP13] and [Kni16]. Such manifolds do not have conjugate points and, for every $x \in X$, the exponential map $\exp_x : T_x X \to X$ is a diffeomorphism. (See e.g [Kni02] on basic geometric and dynamical properties of spaces without conjugate points.) The absence of conjugate points in X allows to define Busemann functions associated to geodesic rays $\gamma_v : [0, \infty) \to X$ with $\gamma_v'(0) = v$. These functions are of central importance in our paper and are given by

$$b_v(y) = \lim_{t \to \infty} (d(y, \gamma_v(t)) - t).$$

The level sets of these functions are called *horospheres* and can be viewed as spheres with center at infinity.

For any $v \in T_x^1 X$ and $r > 0$, let $A(v, r)$ denote the Jacobian of the map $v \mapsto \exp_x(rv)$. The definition of harmonicity given in the Introduction is equivalent to the fact that this Jacobian does not depend on v, i.e. there is a function A on $(0, \infty)$ such that $A(v, r) = A(r)$ for all $v \in T^1 X$. See [Wil93, p. 224] for the equivalence of this property with the property given in the Introduction. The function A is called the *density function* of the harmonic manifold.

For $x \in X$, let d_x denote the distance function from the point x, i.e. $d_x(y) = d(x, y)$. A function f on X is said to be *radial* around a point x of X if f is constant on geodesic spheres centered at x. For each $x \in X$, we can define a radialization operator M_x, defined for a continuous function f on X by

$$(M_x f)(z) = \int_{S(x,r)} f(y) d\sigma^r(y)$$

where $S(x, r)$ denotes the geodesic sphere around x of radius $r = d(x, z)$, and σ^r denotes surface area measure on this sphere (induced from the metric on X), normalized to have mass one. The operator M_x maps continuous functions to functions radial around x, and is formally self-adjoint, meaning

$$\int_X (M_x f)(z) h(z) dvol(z) = \int_X f(z) (M_x h)(z) dvol(z).$$

for all continuous functions f, h with compact support. Introducing polar coordinates around x this follows easily from

$$\int_X (M_x f)(z) h(z) dvol(z) = \int_0^\infty \int_{T_x^1 X} f(\gamma_v(r)) d\theta_x(v) \int_{T_x^1 X} h(\gamma_w(r)) d\theta_x(w) A(r) dr,$$

where θ_x is the normalized canonical measure on the unit tangent space $T_x^1 X$ induced by the Riemannian metric and $\gamma_v : \mathbb{R} \to X$ is the geodesic satisfying $\gamma_v'(0) = v$.

Using these concepts, we have the following equivalent conditions for harmonicity:

(1) For any $x \in X$, Δd_x is radial around x.
(2) The Laplacian $\Delta = \mathrm{div} \circ \nabla$ commutes with all the radialization operators M_x, i.e. $M_x \Delta u = \Delta M_x u$ for all smooth functions u on X and all $x \in X$.
(3) For any smooth function u radial around any $x \in X$ the function Δu is radial around x, as well.

Let us now discuss basic properties of the density function $A(r)$ of a non-compact harmonic manifold. The function $A(r)$ is increasing in r, and the quantity $(A'/A)(r) \geq 0$ equals the mean curvature of geodesic spheres $S(x,r)$ of radius r, which decreases monotonically as $r \to \infty$ (see [RS03, Corollary 2.1, Proposition 2.2] and [Kni02, Section 1.2]). Furthermore, the mean curvature $(A'/A)(r)$ of the geodesic sphere $S(x,r)$ at a point $z \in S(x,r)$ equals $\Delta d_x(z)$, hence we have

$$\Delta d_x = \frac{A'}{A} \circ d_x.$$

The limit $\lim_{r\to\infty}(A'/A)(r)$ is equal to the mean curvature $h \geq 0$ of horospheres. Therefore, all harmonic manifolds are in particular *asymptotically harmonic*, meaning they are manifolds without conjugate points such that all horospheres have the same constant mean curvature.

Using the density function $A(r)$, harmonic manifolds are of purely exponential volume growth if and only if there exist constants $C > 1$, $h > 0$ such that we have for all $R > 1$

$$\frac{1}{C} e^{hR} \leq A(R) \leq C e^{hR}.$$

In this particular case it turns out that the constant $h > 0$ agrees with the mean curvature of the horospheres.

Let us finish this section by discussing specific properties of non-compact simply connected *harmonic manifolds* (X,g) of *purely exponential volume growth* as defined in (1). As mentioned in the introduction, in this setting purely exponential volume growth, Anosov geodesic flow and Gromov hyperbolicity are equivalent properties (see [Kni12]).

Let $\overline{X} = X \cup \partial X$ be as defined in the previous section. For each $x \in X$, we introduce the following bijective map $pr_x : B_1(x) \to \bar{X}$, where $B_1(x) \subset T_x X$ is the closed ball of radius 1:

$$pr_x(v) = \begin{cases} \gamma_v(\infty) & \text{if } \|v\| = 1, \\ \exp_x\left(\frac{1}{1-\|v\|} v\right) & \text{if } \|v\| < 1. \end{cases}$$

Then the map pr_x is a homeomorphism (where $\overline{X}$ is equipped with the cone topology) (see [KP16] section 3 and 4).

Since the horospheres are the footpoint projections of the stable manifolds of the geodesic flow, we have the following convergence property of asymptotic geodesic starting from the same horosphere in the case of Anosov geodesic flow:

Lemma 2.1. *Given $\xi = \gamma_v(\infty) \in \partial X$ and $x, y \in X$ such that $b_v(x) = b_v(y) = 0$, and geodesics $\gamma_1, \gamma_2 : [0, +\infty) \to X$ such that $\gamma_1(0) = x, \gamma_2(0) = y$ and $\gamma_1(\infty) = \gamma_2(\infty) = \xi$, we have that $d(\gamma_1(t), \gamma_2(t)) \to 0$ as $t \to \infty$.*

Proof: Let $v_1 = \gamma_1'(0), v_2 = \gamma_2'(0)$, then by hypothesis v_1, v_2 lie in the same strong stable manifold for the Anosov geodesic flow, hence $d_{T^1X}(\phi_t(v_1), \phi_t(v_2)) \to 0$ as $t \to +\infty$ (where d_{T^1X} denotes the distance on T^1X induced by the Sasaki metric and $(\phi_t)_{t\in\mathbb{R}}$ denotes the geodesic flow), hence $d(\gamma_1(t), \gamma_2(t)) \to 0$ as $t \to +\infty$, since the canonical projection $\pi : T^1X \to X$ is 1-Lipschitz. $\diamond$

Using this fact we define Busemann functions alternatively with respect to boundary points as follows:

Lemma 2.2. *Let (X, g) be a simply connected harmonic manifold of purely exponential volume growth and $x \in X$ and $\xi \in \partial X$. Then the* Busemann function $B_{\xi,x} : X \to \mathbb{R}$ *is defined by*

$$B_{\xi,x}(y) = \lim_{t\to\infty} (d(y, \gamma(t)) - d(x, \gamma(t)))$$

where $\gamma : [0, \infty) \to X$ is a geodesic ray with $\gamma(\infty) = \xi$. This definition does not depend on the choice of γ.

Proof: Let $\gamma_0 : [0, \infty) \to X$ be the geodesic ray with $\gamma_0(0) = x$ and $\gamma_0(\infty) = \xi$. Let $v = \gamma_0'(0)$. Then there exists $t_0 \in \mathbb{R}$ such that we have

$$d(\gamma_0(t + t_0), \gamma(t)) \to 0 \quad \text{for } t \to \infty,$$

and we have

$$\begin{aligned} d(y, \gamma(t)) - d(x, \gamma(t)) = d(y, \gamma_0(t + t_0)) &+ (d(y, \gamma(t)) - d(y, \gamma_0(t + t_0))) \\ &- d(x, \gamma_0(t + t_0)) - (d(x, \gamma(t)) - d(x, \gamma_0(t + t_0))). \end{aligned}$$

Since

$$|d(z, \gamma(t)) - d(z, \gamma_0(t + t_0))| \le d(\gamma(t), \gamma_0(t + t_0)) \to 0 \quad \text{for } t \to \infty, \tag{2}$$

we obtain

$$\begin{aligned} \lim_{t\to\infty} (d(y, \gamma(t)) - d(x, \gamma(t))) = \lim_{t\to\infty} (d(y, \gamma_0(t + t_0)) - d(x, \gamma_0(t + t_0))) = \\ \lim_{t\to\infty} (d(y, \gamma_0(t + t_0)) - (t + t_0)) = b_v(q). \end{aligned}$$

This shows the independence of the limit of the choice of geodesic ray. $\diamond$

The level sets of $B_{\xi,x}$ are called *horospheres* centered at ξ and their mean curvatures agree with $\Delta B_{\xi,x}$ for all $\xi \in \partial X, x \in X$. Since they have the same constant mean curvature $h \ge 0$, we have

$$\Delta B_{\xi,x} = h.$$

In the case of purely exponential volume growth the constant h is positive. The *Busemann cocycle* $B : X \times X \times \partial X \to \mathbb{R}$ is defined by

$$B(x, y, \xi) := B_{\xi,y}(x),$$

and it is easy to see that it satisfies the following cocycle property:

$$B(x, z, \xi) = B(x, y, \xi) + B(y, z, \xi).$$

Since (X, g) is a Gromov hyperbolic space by [Kni12], it is equipped with the Gromov product defined in section 2.1. It will be necessary for our purposes to work however with the modified version of the Gromov product given by the following lemma:

Lemma 2.3. *Let (X, g) be a simply connected harmonic manifold of purely exponential volume growth. For $\xi, \eta \in \partial X$, and $\gamma_1, \gamma_2 : [0, \infty) \to X$ any two geodesic rays with $\gamma_1(\infty) = \xi$ and $\gamma_2(\infty) = \eta$, the limit*

$$\lim_{s,t\to\infty} (\gamma_1(s)|\gamma_2(t))_x,$$

exists and is independent of the choice of the geodesic rays γ_1, γ_2.

Proof: We first assume $\xi \neq \eta$. Since X is Gromov hyperbolic, there exists a geodesic $\gamma : \mathbb{R} \to X$ with $\gamma(-\infty) = \xi$ and $\gamma(\infty) = \eta$ (see, e.g., [DK18, Lemma 11.83]). Using Lemma 2.1, we conclude that there exist $s_0, t_0 \in \mathbb{R}$ such that

$$d(\gamma_1(s), \gamma(-s + s_0)) \to 0 \quad \text{as } s \to \infty$$

and

$$d(\gamma_2(t), \gamma(t + t_0)) \to 0 \quad \text{as } t \to \infty.$$

Using these limits and similar arguments as in the proof of Lemma 2.2 (in particular (2)), we derive

$$\begin{aligned}
\lim_{s,t\to\infty} (\gamma_1(s)|\gamma_2(t))_x &= \lim_{s,t\to\infty} \frac{1}{2}(d(\gamma_1(s), x) + d(\gamma_2(t), x) - d(\gamma_1(s), \gamma_2(t))) \\
&= \lim_{s,t\to\infty} \frac{1}{2}(d(\gamma(-s + s_0), x) + d(\gamma(t + t_0), x) - d(\gamma(-s + s_0), \gamma(t + t_0))) \\
&= \frac{1}{2}(\lim_{s\to\infty} (d(\gamma(-s + s_0), x) - (s - s_0))) + \frac{1}{2}(\lim_{t\to\infty} (d(\gamma(t + t_0), x) - (t + t_0))) \\
&= \frac{1}{2}(B_{\xi,\gamma(0)}(x) + B_{\eta,\gamma(0)}(x)).
\end{aligned}$$

Next we assume $\xi = \eta$. Let $\gamma, \gamma_1, \gamma_2 : [0, \infty) \to X$ be geodesic rays with $\gamma(0) = x$ and $\gamma(\infty) = \gamma_1(\infty) = \gamma_2(\infty) = \xi$. Again, we can find $t_1, t_2 \in \mathbb{R}$ such that for $i \in \{1, 2\}$,

$$d(\gamma_i(t), \gamma(t + t_i)) \to 0 \quad \text{as } t \to \infty.$$

Using these limits again we derive

$$\begin{aligned}
\lim_{s,t\to\infty} (\gamma_1(s)|\gamma_2(t))_x &= \lim_{s,t\to\infty} \frac{1}{2}(d(\gamma_1(s), x) + d(\gamma_2(t), x) - d(\gamma_1(s), \gamma_2(t))) \\
&= \lim_{s,t\to\infty} \frac{1}{2}(d(\gamma(s + t_1), x) + d(\gamma(t + t_2), x) - d(\gamma(s + t_1), \gamma(t + t_2))) \\
&= \lim_{s,t\to\infty} \frac{1}{2}(s + t_1 + t + t_2 - |s + t_1 - (t + t_2)|) = \infty.
\end{aligned}$$

◇

For the rest of the article, by the Gromov product we will mean the limit given by the above lemma, and denote it by the same symbol $(.|.)_x$. We remark that this definition of Gromov product may not agree with the definition given for general Gromov hyperbolic spaces in section 2.1 as a liminf, however the two definitions do agree in the case of CAT(-1) spaces, in particular if the manifold X has strictly negative curvature.

The main reason for introducing this modified version of the Gromov product is that we then have the following relation between Busemann functions and the Gromov product in our setting (it also holds in any CAT(-1) space):

Lemma 2.4. *Let X be a noncompact, simply connected harmonic manifold of purely exponential volume growth. For $x \in X$ and $\eta \in \partial X$, let $\gamma_{x,\eta} : [0,\infty) \to X$ be a geodesic ray with $\gamma_{x,\eta}(0) = x$ and $\gamma_{x,\eta}(\infty) = \eta$. Then we have for all $\xi \in \partial X$:*

$$\lim_{r\to\infty} \left(B_{\xi,x}(\gamma_{x,\eta}(r)) - r\right) = -2(\xi|\eta)_x.$$

Proof: Let $\alpha : [0,\infty) \to X$ be a geodesic ray with $\alpha(0) = x$ and $\alpha(\infty) = \xi$. Then by the previous Lemma, the double limit

$$\lim_{s,r\to\infty} d(\alpha(s), \gamma_{x,\eta}(r)) - (r+s)$$

exists and equals $-2(\xi|\eta)_x$. Since the double limit exists, it can be evaluated as an iterated limit, so we have:

$$-2(\xi|\eta)_x = \lim_{r\to\infty} \left(\lim_{s\to\infty} d(\alpha(s), \gamma_{x,\eta}(r)) - (r+s) \right)$$

Now for a fixed r we have $\lim_{s\to\infty}(d(\alpha(s), \gamma_{x,\eta}(r)) - (r+s)) = B_{\xi,x}(\gamma_{x,\eta}(r)) - r$, so substituting this in the previous equation gives the result. $\diamond$

Finally, we define the family of visibility measures λ_x on harmonic manifolds (X, g) of purely exponential volume growth. For $x \in X$, let θ_x denote the normalized canonical measure on $T^1_x X$ induced by the Riemannian metric and λ_x be the push forward of θ_x to the boundary ∂X under pr_x. The *visibility measures* λ_x are pairwise absolutely continuous with Radon-Nykodym derivative given by

$$\frac{d\lambda_y}{d\lambda_x}(\xi) = e^{-hB_{\xi,x}(y)}. \tag{3}$$

This result was shown in [KP16, Theorem 1.4] in the more general setting of asymptotically harmonic manifolds of purely exponential volume growth with curvature tensor bounds $\|R\|_\infty \le R_0$, $\|\nabla R\|_\infty \le R_0'$ for some $R_0, R_0' > 0$. These curvature tensor bounds are satisfied for harmonic manifolds by [Bes78, Propositions 6.57 and 6.68].

3. Radial and horospherical parts of the Laplacian

Let X be a non-compact simply connected harmonic manifold. Let $h \ge 0$ denote the mean curvature of horospheres in X, let $\rho = \frac{1}{2}h$, and let $A : (0,\infty) \to \mathbb{R}$ denote the density function of X.

Lemma 3.1. *For f a C^2 function on X and u a C^∞ function on $\mathbb{R}$, we have*

$$\Delta(u \circ f) = (u'' \circ f)|\nabla f|^2 + (u' \circ f)\Delta f.$$

Proof: Let γ be a geodesic, then $(u \circ f \circ \gamma)'(t) = (u' \circ f)(\gamma(t)) < \nabla f, \gamma'(t) >$, so

$$(u \circ f \circ \gamma)''(t) = (u'' \circ f)(\gamma(t)) < \nabla f, \gamma'(t) >^2 + (u' \circ f)(\gamma(t)) < \nabla_{\gamma'} \nabla f, \gamma'(t) > .$$

Now let $\{e_i\}$ be an orthonormal basis of T_xX, and let γ_i be geodesics with $\gamma_i'(0) = e_i$. Then

$$\begin{aligned}
\Delta(u \circ f)(x) &= \sum_{i=1}^{n} < \nabla_{e_i} \nabla(u \circ f), e_i > \\
&= \sum_{i=1}^{n} (u \circ f \circ \gamma_i)''(0) \\
&= (u'' \circ f)(x) \sum_{i=1}^{n} < \nabla f, e_i >^2 + (u' \circ f)(x) \sum_{i=1}^{n} < \nabla_{e_i} \nabla f, e_i > \\
&= (u'' \circ f)(x) |\nabla f(x)|^2 + (u' \circ f)(x) \Delta f(x).
\end{aligned}$$

$\diamond$

Any C^∞ function on X radial around $x \in X$ is of the form $f = u \circ d_x$ for some even C^∞ function u on $\mathbb{R}$, where d_x denotes the distance function from the point x, while any C^∞ function which is constant on horospheres at $\xi \in \partial X$ is of the form $f = u \circ B_{\xi,x}$ for some C^∞ function u on $\mathbb{R}$. The following proposition says that the Laplacian Δ leaves invariant these spaces of functions, and describes the action of the Laplacian on these spaces:

Proposition 3.2. *Let $x \in X, \xi \in \partial X$.*

(1) For u a C^∞ function on $(0, \infty)$,

$$\Delta(u \circ d_x) = (L_R u) \circ d_x$$

where L_R is the differential operator on $(0, \infty)$ defined by

$$L_R = \frac{d^2}{dr^2} + \frac{A'(r)}{A(r)} \frac{d}{dr}$$

(2) For u a C^∞ function on $\mathbb{R}$,

$$\Delta(u \circ B_{\xi,x}) = (L_H u) \circ B_{\xi,x}$$

where L_H is the differential operator on $\mathbb{R}$ defined by

$$L_H = \frac{d^2}{dt^2} + 2\rho \frac{d}{dt}$$

Proof: Noting that $|\nabla d_x| = 1, |\nabla B_{\xi,x}| = 1$, and $\Delta d_x = (A'/A) \circ d_x, \Delta B_{\xi,x} = 2\rho$, the Proposition follows immediately from the previous Lemma. $\diamond$

Accordingly, we call the differential operators L_R and L_H the *radial and horospherical parts of the Laplacian* respectively. It follows from the above proposition that a function $f = u \circ d_x$ radial around x is an eigenfunction of Δ with eigenvalue σ if and only if u is an eigenfunction of L_R with eigenvalue σ. Similarly, a function $f = u \circ B_{\xi,x}$ constant on horospheres at ξ is an eigenfunction of Δ with eigenvalue σ if and only if u is an eigenfunction of L_H with eigenvalue σ. In particular, we have the following:

Proposition 3.3. *Let $x \in X, \xi \in \partial X$. Then for any $\lambda \in \mathbb{C}$, the function*

$$f = e^{(i\lambda-\rho)B_{\xi,x}}$$

is an eigenfunction of the Laplacian with eigenvalue $-(\lambda^2+\rho^2)$ satisfying $f(x) = 1$.

Proof: This follows from the fact that the function $u(t) = e^{(i\lambda-\rho)t}$ on $\mathbb{R}$ is an eigenfunction of L_H with eigenvalue $-(\lambda^2+\rho^2)$, and $B_{\xi,x}(x) = 0$ gives $f(x) = 1$. $\diamond$

4. Analysis of radial functions

As we saw in the previous section, finding radial eigenfunctions of the Laplacian amounts to finding eigenfunctions of its radial part L_R. When X is a rank one symmetric space G/K, or more generally a harmonic NA group, then the volume density function is of the form $A(r) = C\left(\sinh\left(\frac{r}{2}\right)\right)^p \left(\cosh\left(\frac{r}{2}\right)\right)^q$, for a constant $C > 0$ and integers $p, q \geq 0$, and so the radial part $L_R = \frac{d^2}{dr^2} + (A'/A)\frac{d}{dr}$ falls into the general class of *Jacobi operators*

$$L_{\alpha,\beta} = \frac{d^2}{dr^2} + ((2\alpha+1)\coth r + (2\beta+1)\tanh r)\frac{d}{dr}$$

for which there is a detailed and well known harmonic analysis in terms of eigenfunctions (called *Jacobi functions*) [Koo84]. For a general harmonic manifold X, the explicit form of the density function A is not known, so it is unclear whether the radial part L_R is a Jacobi operator. However, there is a harmonic analysis, based on hypergroups ([Che74], [Che79], [Tri81], [Tri97b], [Tri97a], [BX95], [Xu94]), for more general second-order differential operators on $(0,\infty)$ of the form

$$L = \frac{d^2}{dr^2} + \frac{A'(r)}{A(r)}\frac{d}{dr} \tag{4}$$

where $A : [0,\infty) \to [0,\infty)$ is a function satisfying certain hypotheses which allow one to endow $[0,\infty)$ with a hypergroup structure, called a *Chebli-Trimeche hypergroup.* We first recall some basic facts about Chebli-Trimeche hypergroups, and then show that the density function of a harmonic manifold satisfies the hypotheses required in order to apply this theory.

4.1. Chebli-Trimeche hypergroups.

A hypergroup $(K,*)$ is a locally compact Hausdorff space K such that the space $M^b(K)$ of finite Borel measures on K is endowed with a product $(\mu,\nu) \mapsto \mu * \nu$ turning it into an algebra with unit, and K is endowed with an involutive homeomorphism $x \in K \mapsto \tilde{x} \in K$, such that the product and the involution satisfy certain natural properties (see [BH95] Chapter 1 for the precise definition). A motivating example relevant to the following is the algebra of finite radial measures on a noncompact rank one symmetric space G/K under convolution; as radial measures can be viewed as measures on $[0,\infty)$, this endows $[0,\infty)$ with a hypergroup structure (with the involution being the identity). It turns out that this hypergroup structure on $[0,\infty)$ is a special case of a general class of hypergroup structures on $[0,\infty)$ called *Sturm-Liouville hypergroups* (see [BH95], section 3.5). These hypergroups arise from Sturm-Liouville boundary problems on $(0,\infty)$. We will be interested in a particular class of Sturm-Liouville hypergroups called *Chebli-Trimeche hypergroups.* These arise as follows (we refer to [BH95], section 3.5, for proofs of statements below):

A *Chebli-Trimeche function* is a continuous function $A : [0, \infty) \to [0, \infty)$ which is C^∞ and positive on $(0, \infty)$ and satisfies the following conditions:

(H1) A is increasing, and $A(r) \to +\infty$ as $r \to +\infty$.

(H2) A'/A is decreasing, and $\rho = \frac{1}{2} \lim_{r\to\infty} A'(r)/A(r) > 0$.

(H3) For $r > 0$, $A(r) = r^{2\alpha+1} B(r)$ for some $\alpha > -1/2$ and some even, C^∞ function B on $\mathbb{R}$ such that $B(0) = 1$.

Let L be the differential operator on $C^2(0, \infty)$ defined by equation (4), where A satisfies conditions (H1)-(H3) above. Define the differential operator l on $C^2((0, \infty)^2)$ by

$$\begin{aligned} l[u](x, y) &= (L)_x u(x, y) - (L)_y u(x, y) \\ &= \left(u_{xx}(x, y) + \frac{A'(x)}{A(x)} u_x(x, y) \right) - \left(u_{yy}(x, y) + \frac{A'(y)}{A(y)} u_y(x, y) \right) \end{aligned}$$

For $f \in C^2([0, \infty))$ denote by u_f the solution of the hyperbolic Cauchy problem

$$\begin{aligned} l[u_f] &= 0, \\ u_f(x, 0) = u_f(0, x) &= f(x), \\ (u_f)_y(x, 0) &= 0, \\ (u_f)_x(0, y) &= 0 \quad \text{for } x, y \in [0, \infty). \end{aligned}$$

For $x \in [0, \infty)$, let ϵ_x denote the Dirac measure of mass one at x. Then for all $x, y \in [0, \infty)$, there exists a probability measure on $[0, \infty)$ denoted by $\epsilon_x * \epsilon_y$ such that

$$\int_0^\infty f d(\epsilon_x * \epsilon_y) = u_f(x, y)$$

for all even, C^∞ functions f on $\mathbb{R}$. We have $\epsilon_x * \epsilon_y = \epsilon_y * \epsilon_x$ for all x, y, and the product $(\epsilon_x, \epsilon_y) \mapsto \epsilon_x * \epsilon_y$ extends to a product on all finite measures on $[0, \infty)$ which turns $[0, \infty)$ into a commutative hypergroup $([0, \infty), *)$ (with the involution being the identity), called the *Chebli-Trimeche hypergroup* associated to the function A ([BH95], section 3.5).

A measure μ on a commutative hypergroup $(K, *)$ is called a *Haar measure* for the hypergroup if

$$\int_K f d(\epsilon_x * \mu) = \int_K f d\mu$$

for all $f \in C_c(K)$ and all $x \in K$. Any commutative hypergroup has a Haar measure ([BH95], Theorem 1.3.15), which in the case of the Chebli-Trimeche hypergroup $([0, \infty), *)$ is given by the measure $A(r)dr$ on $[0, \infty)$.

For a commutative hypergroup K with a Haar measure dk, a Fourier analysis can be carried out analogous to the Fourier analysis on locally compact abelian groups ([BH95], section 2.2). There is a dual space $\hat{K}$ of characters, which are bounded multiplicative functions on the hypergroup $\chi : K \to \mathbb{C}$ satisfying $\chi(\tilde{x}) = \overline{\chi(x)}$, where multiplicative means that

$$\int_K \chi d(\epsilon_x * \epsilon_y) = \chi(x)\chi(y)$$

for all $x, y \in K$. For $f \in L^1(K)$, the Fourier transform of f is the function $\hat{f}$ on $\hat{K}$ defined by

$$\hat{f}(\chi) = \int_K f\overline{\chi} dk.$$

The Levitan-Plancherel Theorem ([BH95], Theorem 2.2.13) states that there is a measure $d\chi$ on $\hat{K}$ called the Plancherel measure, such that the mapping $f \mapsto \hat{f}$ extends from $L^1(K) \cap L^2(K)$ to an isometry from $L^2(K)$ onto $L^2(\hat{K})$. The inverse Fourier transform of a function $\sigma \in L^1(\hat{K})$ is the function $\check{\sigma}$ on K defined by

$$\check{\sigma}(k) = \int_{\hat{K}} \sigma(\chi)\chi(k) d\chi.$$

The Fourier inversion theorem ([BH95], Theorem 2.2.36) then states that if $f \in L^1(K) \cap C(K)$ is such that $\hat{f} \in L^1(\hat{K})$, then $f = (\hat{f})\check{}$, i.e.

$$f(x) = \int_{\hat{K}} \hat{f}(\chi)\chi(x) d\chi$$

for all $x \in K$.

For the Chebli-Trimeche hypergroup, it turns out that the multiplicative functions on the hypergroup are given precisely by eigenfunctions of the operator L. For any $\lambda \in \mathbb{C}$, the equation

$$Lu = -(\lambda^2 + \rho^2)u \tag{5}$$

has a unique solution ϕ_λ on $(0, \infty)$ which extends continuously to 0 and satisfies $\phi_\lambda(0) = 1$ (note that the coefficient A'/A of the operator L is singular at $r = 0$ so existence of a solution continuous at 0 is not immediate). The function ϕ_λ extends to a C^∞ even function on $\mathbb{R}$. Since equation (5) reads the same for λ and $-\lambda$, by uniqueness we have $\phi_\lambda = \phi_{-\lambda}$.

The multiplicative functions on $[0, \infty)$ are then exactly the functions $\phi_\lambda, \lambda \in \mathbb{C}$. The functions ϕ_λ are bounded if and only if $|\operatorname{Im}\lambda| \leq \rho$. Furthermore, the involution on the hypergroup being the identity, the characters of the hypergroup are real-valued, which occurs for ϕ_λ if and only if $\lambda \in \mathbb{R} \cup i\mathbb{R}$. Thus the dual space of the hypergroup is given by ([BH95], Theorem 3.5.50)

$$\hat{K} = \{\phi_\lambda | \lambda \in [0, \infty) \cup [0, i\rho]\}$$

which we identify with the set $\Sigma = [0, \infty) \cup [0, i\rho] \subset \mathbb{C}$.

The hypergroup Fourier transform of a function $f \in L^1([0, \infty), A(r)dr)$ is given by

$$\hat{f}(\lambda) = \int_0^\infty f(r)\phi_\lambda(r)A(r)dr$$

for $\lambda \in \Sigma$ (when the hypergroup arises from convolution of radial measures on a rank one symmetric space G/K, then this is the well-known Jacobi transform [Koo84]). The Levitan-Plancherel and Fourier inversion theorems for the hypergroup give the existence of a Plancherel measure σ on Σ such that the Fourier transform defines an isometry from $L^2([0, \infty), A(r)dr)$ onto $L^2(\Sigma, \sigma)$, and, for any function $f \in L^1([0, \infty), A(r)dr) \cap C([0, \infty))$ such that $\hat{f} \in L^1(\Sigma, \sigma)$, we have

$$f(r) = \int_\Sigma \hat{f}(\lambda)\phi_\lambda(r) d\sigma(\lambda)$$

for all $r \in [0, \infty)$.

In [BX95], it is shown that under certain extra conditions on the function A, the support of the Plancherel measure is $[0,\infty)$ and the Plancherel measure is absolutely continuous with respect to Lebesgue measure $d\lambda$ on $[0,\infty)$, given by

$$d\sigma(\lambda) = C_0|c(\lambda)|^{-2}d\lambda$$

where $C_0 > 0$ is a constant, and c is a certain complex function on $\mathbb{C} - \{0\}$. The required conditions on A are as follows:

Making the change of dependent variable $v = A^{1/2}u$, equation (5) becomes

$$v''(r) = (G(r) - \lambda^2)v(r) \tag{6}$$

where the function G is defined by

$$G(r) = \frac{1}{4}\left(\frac{A'(r)}{A(r)}\right)^2 + \frac{1}{2}\left(\frac{A'}{A}\right)'(r) - \rho^2 \tag{7}$$

If the function G tends to 0 fast enough near infinity, then it is reasonable to expect that equation (6) above has two linearly independent solutions asymptotic to exponentials $e^{\pm i\lambda r}$ near infinity. Bloom-Xu show that this is indeed the case [BX95] under the following hypothesis on the function G:

(H4) For some $r_0 > 0$, we have

$$\int_{r_0}^{\infty} r|G(r)|dr < +\infty$$

and G is bounded on $[r_0,\infty)$.

Under hypothesis (H4), for any $\lambda \in \mathbb{C} - \{0\}$, there are unique solutions $\Phi_\lambda, \Phi_{-\lambda}$ of equation (5) on $(0,\infty)$ which are asymptotic to exponentials near infinity [BX95],

$$\Phi_{\pm\lambda}(r) = e^{(\pm i\lambda - \rho)r}(1 + o(1)) \quad \text{as } r \to +\infty$$

The solutions $\Phi_\lambda, \Phi_{-\lambda}$ are linearly independent, so, since $\phi_\lambda = \phi_{-\lambda}$, there exists a function c on $\mathbb{C} - \{0\}$ such that

$$\phi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda}$$

for all $\lambda \in \mathbb{C} - \{0\}$. We will call this function the c-function of the hypergroup. We remark that if the hypergroup $([0,\infty),*)$ is the one arising from convolution of radial measures on a noncompact rank one symmetric space G/K, then this function agrees with Harish-Chandra's c-function only on the half-plane $\{\operatorname{Im}\lambda \leq 0\}$ and not on all of $\mathbb{C}$.

If we furthermore assume the hypothesis $|\alpha| \neq 1/2$, then Bloom-Xu show that the function c is non-zero for $\operatorname{Im}\lambda \leq 0, \lambda \neq 0$, and prove the following estimates:

There exist constants $C, K > 0$ such that

$$\frac{1}{C}|\lambda| \leq |c(\lambda)|^{-1} \leq C|\lambda|, \qquad |\lambda| \leq K$$
$$\frac{1}{C}|\lambda|^{\alpha+\frac{1}{2}} \leq |c(\lambda)|^{-1} \leq C|\lambda|^{\alpha+\frac{1}{2}}, \quad |\lambda| \geq K$$

Moreover they prove the following inversion formula: for any even function $f \in C_c^\infty(\mathbb{R})$,

$$f(r) = C_0 \int_0^\infty \hat{f}(\lambda)\phi_\lambda(r)|c(\lambda)|^{-2} d\lambda$$

where $C_0 > 0$ is a constant.

It follows that the Plancherel measure σ of the hypergroup is supported on $[0,\infty)$, and absolutely continuous with respect to Lebesgue measure, with density given by $C_0|c(\lambda)|^{-2}$. Bloom-Xu also show that the c-function is holomorphic on the half-plane $\{\operatorname{Im}\lambda < 0\}$.

4.2. **The density function of a harmonic manifold.** Throughout this section and the next, we denote by X a simply connected, n-dimensional harmonic manifold of purely exponential volume growth.

Let A be the density function of X. We check that A is a Chebli-Trimeche function, so that we obtain a commutative hypergroup $([0,\infty),*)$, and that the conditions of Bloom-Xu are met so that the Plancherel measure is given by $C_0|c(\lambda)|^{-2}d\lambda$ on $[0,\infty)$.

The function $A(r)$ equals, up to a constant factor, the volume of geodesic spheres $S(x,r)$, which is increasing in r and tends to infinity as r tends to infinity, so condition (H1) is satisfied. As stated in section 2.2, the function $A'(r)/A(r)$ equals the mean curvature of geodesic spheres $S(x,r)$, which decreases monotonically to a limit $h = 2\rho$ which is positive (and equals the mean curvature of horospheres), so condition (H2) is satisfied.

Fixing a point $x \in X$, for $r > 0$, the density function $A(r)$ is given by the Jacobian of the map $\phi : v \mapsto \exp_x(rv)$ from the unit tangent sphere $T_x^1 X$ to the geodesic sphere $S(x,r)$. Let T be the map $v \mapsto rv$ from the unit tangent sphere $T_x^1 X$ to the tangent sphere of radius r, $T_x^r X \subset T_x M$, then $\phi = \exp_x \circ T$, so the Jacobian of ϕ is given by the product of the Jacobians of T and $\exp_x$, hence

$$A(r) = r^{n-1}B(r)$$

where the function B is given by

$$B(r) = \det(D\exp_x)_{rv}$$

where v is any fixed vector in $T_x^1 X$. Since B is independent of the choice of v, in particular is the same for vectors v and $-v$, the function B is even, and C^∞ on $\mathbb{R}$ with $B(0) = 1$. Thus condition (H3) holds for the function A, with $\alpha = (n-2)/2$.

The density function A is thus a Chebli-Trimeche function, so we obtain a hypergroup structure on $[0,\infty)$, which we call the *radial hypergroup* of the harmonic manifold X (the reason for this terminology will become clear from the the following sections).

We proceed to check that condition (H4) is satisfied. For this we will need the following theorem of Nikolayevsy:

Theorem 4.1. [Nik05] *The density function of a harmonic manifold is an exponential polynomial, i.e. a function of the form*

$$A(r) = \sum_{i=1}^{k} (p_i(r)\cos(\beta_i r) + q_i(r)\sin(\beta_i r))e^{\alpha_i r}$$

where p_i, q_i are polynomials and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \dots, k$.

It will be convenient to rearrange terms and write the density function in the form

$$A(r) = \sum_{i=1}^{l} \sum_{j=0}^{m_i} f_{ij}(r) r^j e^{\alpha_i r} \tag{8}$$

where $\alpha_1 < \alpha_2 < \dots < \alpha_l$, and each f_{ij} is a trigonometric polynomial, i.e. a finite linear combination of functions of the form $\cos(\beta r)$ and $\sin(\beta r)$, $\beta \in \mathbb{R}$, with f_{im_i} not identically zero, for $i = 1, \dots, l$. For an exponential polynomial written in this form, we will call the largest exponent α_l which appears in the exponentials the *exponential degree* of the exponential polynomial.

Lemma 4.2. *With the density function as above, we have $\alpha_l = 2\rho, m_l = 0$ and $f_{l0} = C$ for some constant $C > 0$. Thus the density function is of the form*

$$A(r) = Ce^{2\rho r} + P(r)$$

where P is an exponential polynomial of exponential degree $\delta < 2\rho$.

Proof: Since X has purely exponential volume growth, there exists a constant $C > 1$ such that

$$\frac{1}{C} \leq \frac{A(r)}{e^{2\rho r}} \leq C \tag{9}$$

for all $r \geq 1$. If $\alpha_l < 2\rho$, then $A(r)/e^{2\rho r} \to 0$ as $r \to \infty$, contradicting (9) above, so we must have $\alpha_l \geq 2\rho$. On the other hand, if $\alpha_l > 2\rho$, then since f_{lm_l} is a trigonometric polynomial which is not identically zero, we can choose a sequence r_m tending to infinity such that $f_{lm_l}(r_m) \to \alpha \neq 0$. Then clearly $A(r_m)/e^{2\rho r_m} \to \infty$, again contradicting (9). Hence $\alpha_l = 2\rho$.

Using (8) and $\alpha_l = 2\rho$, we have

$$\frac{A'(r)}{A(r)} - 2\rho = \frac{f'_{lm_l}(r) + o(1)}{f_{lm_l}(r) + o(1)}$$

as $r \to \infty$, thus

$$\begin{aligned} f'_{lm_l}(r) + o(1) &= (f_{lm_l}(r) + o(1)) \left(\frac{A'(r)}{A(r)} - 2\rho \right) \\ &\to 0 \end{aligned}$$

as $r \to \infty$ since f_{lm_l} is bounded and $A'(r)/A(r) - 2\rho \to 0$ as $r \to \infty$. Thus f'_{lm_l} is a trigonometric polynomial which tends to 0 as $r \to \infty$, so it must be identically zero, hence $f_{lm_l} = C$ for some non-zero constant C.

It follows that

$$A(r) = Cr^{m_l} e^{2\rho r}(1 + o(1))$$

as $r \to \infty$. If $m_l \geq 1$ then $A(r)/e^{2\rho r} \to \infty$ as $r \to \infty$, so we must have $m_l = 0$. $\diamond$

Lemma 4.3. *Condition (H4) holds for the density function A, i.e.*

$$\int_{r_0}^{\infty} r|G(r)|dr < +\infty$$

and G is bounded on $[r_0, \infty)$ for any $r_0 > 0$, where

$$G(r) = \frac{1}{4}\left(\frac{A'(r)}{A(r)}\right)^2 + \frac{1}{2}\left(\frac{A'}{A}\right)'(r) - \rho^2$$

Proof: By the previous lemma, $A(r) = Ce^{2\rho r} + P(r)$, where P is an exponential polynomial of exponential degree $\delta < 2\rho$. We then have

$$\begin{aligned}\frac{A'(r)}{A(r)} - 2\rho &= \frac{P'(r) - 2\rho P(r)}{Ce^{2\rho r} + P(r)} \\ &= \frac{Q(r)}{Ce^{2\rho r} + P(r)}\end{aligned}$$

where Q is an exponential polynomial of exponential degree less than or equal to δ. Putting $\kappa = (2\rho - \delta)/2$, it follows that $A'(r)/A(r) - 2\rho = O(e^{-\kappa r})$ as $r \to \infty$. Differentiating, we obtain

$$\begin{aligned}\left(\frac{A'}{A}\right)'(r) &= \frac{(Ce^{2\rho r} + P(r))Q'(r) - Q(r)(2\rho Ce^{2\rho r} + P'(r))}{(Ce^{2\rho r} + P(r))^2} \\ &= \frac{Q_1(r)}{(Ce^{2\rho r} + P(r))^2}\end{aligned}$$

where Q_1 is an exponential polynomial of exponential degree less than or equal to $(2\rho + \delta)$. Since the denominator of the above expression is of the form $ke^{4\rho r} + P_1(r)$ with P_1 an exponential polynomial of exponential degree strictly less than 4ρ, it follows that $(A'/A)'(r) = O(e^{-\kappa r})$ as $r \to \infty$.

Now we can write the function G as

$$G(r) = \frac{1}{4}\left(\frac{A'(r)}{A(r)} - 2\rho\right)\left(\frac{A'(r)}{A(r)} + 2\rho\right) + \frac{1}{2}\left(\frac{A'}{A}\right)'(r)$$

Since $(A'(r)/A(r) + 2\rho)$ is bounded, it follows from the previous paragraph that $G(r) = O(e^{-\kappa r})$ as $r \to \infty$. This immediately implies that condition (H4) holds. $\diamond$

In order to apply the result of Bloom-Xu on the Plancherel measure for the hypergroup, it remains to check that $|\alpha| \neq 1/2$. Since $\alpha = (n-2)/2$, this means $n \neq 3$. Now the Lichnerowicz conjecture holds in dimensions $n \leq 5$ ([Lic44], [Wal48], [Bes78], [Nik05]), i.e. the only harmonic manifolds in such dimensions are the rank one symmetric spaces $X = G/K$, for which as mentioned earlier the Jacobi analysis applies, and the Plancherel measure of the hypergroup is well known to be given by $C_0|\mathbf{c}(\lambda)|^{-2}d\lambda$ where $\mathbf{c}$ is Harish-Chandra's c-function. Thus in our case we may as well assume that X has dimension $n \geq 6$, so that $|\alpha| \neq 1/2$, and we may then apply the results of Bloom-Xu stated in the previous section.

4.3. **The spherical Fourier transform.** Let ϕ_λ denote as in section 4.1 the unique function on $[0, \infty)$ satisfying $L_R \phi_\lambda = -(\lambda^2 + \rho^2)\phi_\lambda$ and $\phi_\lambda(0) = 1$. For $x \in X$ let d_x denote as before the distance function from the point x, $d_x(y) = d(x, y)$. We define the following eigenfunction of Δ radial around x:

$$\phi_{\lambda,x} := \phi_\lambda \circ d_x$$

The uniqueness of ϕ_λ as an eigenfunction of L_R with eigenvalue $-(\lambda^2 + \rho^2)$ and taking the value 1 at $r = 0$ immediately implies the following lemma:

Lemma 4.4. *The function $\phi_{\lambda,x}$ is the unique eigenfunction f of Δ on X with eigenvalue $-(\lambda^2 + \rho^2)$ which is radial around x and satisfies $f(x) = 1$.*

Note that for $\lambda \in \mathbb{R}$, the functions $\phi_{\lambda,x}$ are bounded. Let *dvol* denote the Riemannian volume measure on X.

Definition 4.5. *Let $f \in L^1(X, dvol)$ be radial around the point $x \in X$. We define the spherical Fourier transform of f by*

$$\hat{f}(\lambda) := \int_X f(y)\phi_{\lambda,x}(y)dvol(y)$$

for $\lambda \in \mathbb{R}$.

For f a function on X radial around the point x, let $f = u \circ d_x$ where u is a function on $[0, \infty)$, then evaluating the integral over X in geodesic polar coordinates gives

$$\int_X |f(y)|dvol(y) = \int_0^\infty |u(r)|A(r)dr$$

thus $f \in L^1(X)$ if and only if $u \in L^1([0, \infty), A(r)dr)$. In that case, again integrating in polar coordinates gives

$$\hat{f}(\lambda) = \int_0^\infty u(r)\phi_\lambda(r)A(r)dr = \hat{u}(\lambda)$$

where $\hat{u}$ is the hypergroup Fourier transform of the function u. Moreover $f \in C_c^\infty(X)$ if and only if u extends to an even function on $\mathbb{R}$ such that $u \in C_c^\infty(\mathbb{R})$. Applying the Fourier inversion formula of Bloom-Xu for the radial hypergroup stated in section 4.1 to the function u then leads immediately to the following inversion formula for radial functions:

Theorem 4.6. *Let (X, g) be a simply connected harmonic manifold of purely exponential volume growth and $f \in C_c^\infty(X)$ be radial around the point $x \in X$. Then*

$$f(y) = C_0 \int_0^\infty \hat{f}(\lambda)\phi_{\lambda,x}(y)|c(\lambda)|^{-2}d\lambda$$

for all $y \in X$. Here c denotes the c-function of the radial hypergroup and $C_0 > 0$ is a constant. Moreover, the c-function is holomorphic on the half-plane $\{\operatorname{Im} \lambda < 0\}$.

Proof: As shown in the previous section, all the hypotheses required to apply the inversion formula of Bloom-Xu are satisfied, hence

$$u(r) = C_0 \int_0^\infty \hat{u}(\lambda)\phi_\lambda(r)|c(\lambda)|^{-2}d\lambda.$$

Since $f = u \circ d_x$, this gives

$$\begin{aligned} f(y) &= u(d_x(y)) \\ &= C_0 \int_0^\infty \hat{u}(\lambda)\phi_\lambda(d_x(y))|c(\lambda)|^{-2} d\lambda \\ &= C_0 \int_0^\infty \hat{f}(\lambda)\phi_{\lambda,x}(y)|c(\lambda)|^{-2} d\lambda. \end{aligned}$$

For the holomorphicity of the function c in $\{\operatorname{Im}\lambda < 0\}$ see the proof of Proposition 3.17 in [BX95]. $\diamond$

The Plancherel theorem for the radial hypergroup leads to the following:

Theorem 4.7. *Let (X, g) be a simply connected harmonic manifold of purely exponential volume growth. Let $L^2_x(X, dvol)$ denote the closed subspace of $L^2(X, dvol)$ consisting of those functions in $L^2(X, dvol)$ which are radial around the point x. For $f \in L^1(X, dvol) \cap L^2_x(X, dvol)$, we have*

$$\int_X |f(y)|^2 dvol(y) = C_0 \int_0^\infty |\hat{f}(\lambda)|^2 |c(\lambda)|^{-2} d\lambda$$

The spherical Fourier transform $f \mapsto \hat{f}$ extends to an isometry from $L^2_x(X, dvol)$ onto $L^2([0,\infty), C_0|c(\lambda)|^{-2}d\lambda)$.

Proof: The map $u \mapsto f = u \circ d_x$ defines an isometry of $L^2([0,\infty), A(r)dr)$ onto $L^2_x(X, dvol)$, which maps $L^1([0,\infty), A(r)dr) \cap L^2([0,\infty), A(r)dr)$ onto $L^1(X, dvol) \cap L^2_x(X, dvol)$. The statements of the theorem then follow from the Levitan-Plancherel theorem for the radial hypergroup and from the fact that the Plancherel measure is supported on $[0,\infty)$, given by $C_0|c(\lambda)|^{-2}d\lambda$. $\diamond$

5. Fourier inversion and Plancherel theorem

As before, we assume throughout this section that (X, g) denotes a simply connected harmonic manifold of purely exponential volume growth. We proceed to the analysis of non-radial functions on X. Our definition of Fourier transform will depend on the choice of a basepoint $x \in X$.

Definition 5.1. *Let $x \in X$. For $f \in C^\infty_c(X)$, the Fourier transform of f based at the point x is the function on $\mathbb{C} \times \partial X$ defined by*

$$\tilde{f}^x(\lambda, \xi) = \int_X f(y) e^{(-i\lambda - \rho)B_{\xi,x}(y)} dvol(y)$$

for $\lambda \in \mathbb{C}, \xi \in \partial X$. Here as before $B_{\xi,x}$ denotes the Busemann function at ξ based at x such that $B_{\xi,x}(x) = 0$.

We remark that the Fourier transform defined in the Introduction is the Fourier transform based at $o \in X$. Using the formula

$$B_{\xi,x} = B_{\xi,o} - B_{\xi,o}(x)$$

for points $o, x \in X$, we obtain the following relation between the Fourier transforms based at two different basepoints $o, x \in X$:

$$\tilde{f}^x(\lambda,\xi) = e^{(i\lambda+\rho)B_{\xi,o}(x)}\tilde{f}^o(\lambda,\xi) \tag{10}$$

The key to passing from the inversion formula for radial functions of section 4.3 to an inversion formula for non-radial functions will be a formula expressing the radial eigenfunctions $\phi_{\lambda,x}$ as an integral with respect to $\xi \in \partial X$ of the eigenfunctions $e^{(i\lambda-\rho)B_{\xi,x}}$ (Theorem 5.6). This will be the analogue of the well-known formulae for rank one symmetric spaces G/K and harmonic NA groups expressing the radial eigenfunctions $\phi_{\lambda,x}$ as matrix coefficients of representations of G on $L^2(K/M)$ and NA on $L^2(N)$ respectively.

We start with a basic relation between eigenfunctions of the Laplacian:

Lemma 5.2. *Let $x \in X$ and $\xi \in \partial X$. Then for all $\lambda \in \mathbb{C}$,*

$$\phi_{\lambda,x} = M_x(e^{(i\lambda-\rho)B_{\xi,x}})$$

(where M_x is the radialisation operator around the point x). In particular, $\phi_{\lambda,x}(y)$ is entire in λ for fixed $y \in X$, and is real and positive for λ such that $(i\lambda - \rho)$ is real and positive.

Proof: Since the function $e^{(i\lambda-\rho)B_{\xi,x}}$ is an eigenfunction of the Laplacian Δ with eigenvalue $-(\lambda^2+\rho^2)$ and the operator M_x commutes with Δ, the function $f = M_x(e^{(i\lambda-\rho)B_{\xi,x}})$ is also an eigenfunction of Δ for the eigenvalue $-(\lambda^2+\rho^2)$. Since f is radial around x and $f(x) = 1$, it follows from Lemma 4.4 that $f = \phi_{\lambda,x}$. $\diamond$

The next proposition provides a connection between the Fourier transform and the spherical Fourier transform for radial functions:

Proposition 5.3. *Let $f \in C_c^\infty(X)$ be radial around the point $x \in X$. Then the Fourier transform of f based at x coincides with the spherical Fourier transform,*

$$\tilde{f}^x(\lambda,\xi) = \hat{f}(\lambda)$$

for all $\lambda \in \mathbb{C}, \xi \in \partial X$.

Proof: Let $f = u \circ d_x$ where $u \in C_c^\infty(\mathbb{R})$. By Lemma 5.2 above,

$$\phi_\lambda(r) = \phi_{-\lambda}(r) = \int_{S(x,r)} e^{(-i\lambda-\rho)B_{\xi,x}(y)}d\sigma^r(y)$$

where σ^r is normalized surface area measure on the geodesic sphere $S(x,r)$. Evaluating the integral defining $\tilde{f}^x$ in geodesic polar coordinates centered at x we have

$$\begin{aligned}
\tilde{f}^x(\lambda,\xi) &= \int_0^\infty \int_{S(x,r)} f(y)e^{(-i\lambda-\rho)B_{\xi,x}(y)}d\sigma^r(y)A(r)dr \\
&= \int_0^\infty u(r)\phi_\lambda(r)A(r)dr \\
&= \hat{f}(\lambda)
\end{aligned}$$

$\diamond$

We recall that the visibility measure λ_x on the boundary ∂X with respect to a basepoint $x \in X$, defined in section 2.2, is given by the push-forward $(pr_x)_*\theta_x$ of the normalized canonical measure θ_x on $T^1_x X$ under the map pr_x.

For $\lambda \in \mathbb{C}$ and $x \in X$, define the function $\tilde{\phi}_{\lambda,x}$ on X by

$$\tilde{\phi}_{\lambda,x}(y) = \int_{\partial X} e^{(i\lambda-\rho)B_{\xi,x}(y)} d\lambda_x(\xi)$$

It follows from the above equation that $\tilde{\phi}_{\lambda,x}(y)$ is entire in λ for fixed $y \in X$, and is real and positive for λ such that $(i\lambda - \rho)$ is real and positive. Moreover, by Proposition 3.3, the function $\tilde{\phi}_{\lambda,x}$ is an eigenfunction of the Laplacian Δ with eigenvalue $-(\lambda^2+\rho^2)$, and $\tilde{\phi}_{\lambda,x}(x) = 1$.

Our next aim is to show that $\tilde{\phi}_{\lambda,x}$ is radial around x and, therefore, agrees with the function $\phi_{\lambda,x}$ introduced in Lemma 4.4. We start with a crucial property of non-compact harmonic manifolds without any further assumptions, derived from a result of Szabo that the volume of the intersection of a metric ball $B(x,r_1)$ with a geodesic sphere $S(y,r_2)$ depends only on the radii r_1, r_2 and the distance $d = d(x,y)$ of their centers ([Sza90], Corollary 2.1). We will therefore denote this volume by $v(r_1,r_2,d)$.

Proposition 5.4. *For $v \in T^1_x X$ and $r > 0$, let $b^r_v(y) = d(y,\gamma_v(r)) - r$. Then for every continuous function $\phi : \mathbb{R} \to \mathbb{C}$, the function*

$$F(y) := \int_{T^1_x X} \phi(b^r_v(y)) d\theta_x(v)$$

is radial around x.

Proof: Let $\psi(s) = \phi(s-r)$. Then

$$\phi(b^r_v(y)) = \phi(d(y,\gamma_v(r)) - r) = \psi(d(y,\gamma_v(r)))$$

and

$$F(y) = \int_{T^1_x X} \phi(b^r_v(y)) d\theta_x(v) = \int_{T^1_x X} \psi(d(y,\gamma_v(r))d\theta_x(v). \tag{11}$$

Next, we consider the following expression:

$$\int_{B(x,r)} \psi(d(y,z)) dvol(z) = \int_0^r A(t) \int_{T^1_x X} \psi(d(y,\gamma_v(t))) d\theta_x(v) dt. \tag{12}$$

On the other hand, we have

$$\begin{aligned} \int_{B(x,r)} \psi(d(y,z)) dvol(z) &= \int_0^\infty \int_{B(x,r)\cap S(y,t)} \psi(d(y,z)) d\sigma^{S_y(t)}(z) dt \\ &= \int_0^\infty \int_{B(x,r)\cap S(y,t)} \psi(t) d\sigma^{S_y(t)}(z) dt \\ &= \int_0^\infty vol_{S(y,t)}(B(x,r)\cap S(y,t))\psi(t) dt = \int_0^\infty v(r,t,d(x,y))\psi(t) dt. \end{aligned} \tag{13}$$

Now, we combine (12) and (13) and differentiate with respect to r and obtain

$$A(r) \int_{T^1_x X} \psi(d(y,\gamma_v(r)) d\theta_x(v) = \int_0^\infty \frac{\partial v}{\partial r}(r,t,d(x,y))\psi(t) dt.$$

In view of (11), this implies that

$$F(y) = \frac{1}{A(r)} \int_0^\infty \frac{\partial v}{\partial r}(r,t,d(x,y))\psi(t)dt,$$

which is obviously independent of the position of y within the sphere $S(R,x)$ with $R = d(x,y)$. This shows that the function F is radial around x. $\diamond$

We remark that the above proposition holds without the assumption of purely exponential volume growth. The analogous statement for Busemann functions is obtained via a limiting argument:

Corollary 5.5. *Let $\phi : \mathbb{R} \to \mathbb{C}$ be a continuous function. Then the function*

$$F(y) := \int_{T_x^1 X} \phi(b_v(y)) d\theta_x(v)$$

is a radial function around x.

Proof: Note that we have pointwise convergence $\phi(b_v^r(y)) \to \phi(b_v(y))$ for $r \to \infty$ and, since

$$|b_v^r(y)| \le d(x,y) \quad \text{for all } r \ge 0,$$

we can apply Lebesgue's dominated convergence. $\diamond$

This proposition also holds without the assumption of purely exponential volume growth. As a corollary we obtain the following theorem giving the required formula for the radial eigenfunctions $\phi_{\lambda,x}$ as an integral with respect to $\xi \in \partial X$ of the eigenfunctions $e^{(i\lambda-\rho)B_{\xi,x}}$:

Theorem 5.6. *Let $\lambda \in \mathbb{C}$ and $x \in X$. Then*

$$\phi_{\lambda,x}(y) = \int_{T_x^1 X} e^{(i\lambda-\rho)b_v(y)} d\theta_x(v)$$

for all $y \in X$.

Proof: Both sides are eigenfunctions of the Laplacian Δ with eigenvalue $-(\lambda^2+\rho^2)$. Moreover, both sides assume the value 1 at $y = x$, $\phi_{\lambda,x}$ is radial around x, by definition, and the right hand side is radial by Corollary 5.5 with $\phi(s) = e^{(i\lambda-\rho)s}$. Therefore, both expressions agree by the uniqueness of radial solutions of $\Delta u = -(\lambda^2+\rho^2)u$, $u(x) = 1$. $\diamond$

As with the previous two propositions, the above theorem also holds without the assumption of purely exponential volume growth. We can now prove the Fourier inversion formula:

Theorem 5.7. *Fix a basepoint $o \in X$. Then for $f \in C_c^\infty(X)$ we have*

$$f(x) = C_0 \int_0^\infty \int_{\partial X} \tilde{f}^o(\lambda,\xi) e^{(i\lambda-\rho)B_{\xi,o}(x)} d\lambda_o(\xi) |c(\lambda)|^{-2} d\lambda$$

for all $x \in X$ (where $C_0 > 0$ is a constant).

Proof: Given $f \in C_c^\infty(X)$ and $x \in X$, the function $M_x f$ is in $C_c^\infty(X)$, is radial around the point x and satisfies $(M_x f)(x) = f(x)$. By Theorem 4.6 applied to the function $M_x f$ we have

$$
\begin{aligned}
f(x) = (M_x f)(x) &= C_0 \int_0^\infty \widehat{M_x f}(\lambda) \phi_{\lambda,x}(x) |c(\lambda)|^{-2} d\lambda \\
&= C_0 \int_0^\infty \widehat{M_x f}(\lambda) |c(\lambda)|^{-2} d\lambda
\end{aligned}
$$

since $\phi_{\lambda,x}(x) = 1$. Now using the formal self-adjointness of the operator M_x, Theorem 5.6, the fact that $\phi_{\lambda,x}$ is radial around x and $\phi_{\lambda,x} = \phi_{-\lambda,x}$ we obtain

$$
\begin{aligned}
\widehat{M_x f}(\lambda) &= \int_X (M_x f)(y) \phi_{-\lambda,x}(y) dvol(y) \\
&= \int_X f(y) (M_x \phi_{-\lambda,x})(y) dvol(y) \\
&= \int_X f(y) \phi_{-\lambda,x}(y) dvol(y) \\
&= \int_X f(y) \left(\int_{T_x^1 X} e^{(-i\lambda - \rho) b_v(y)} d\theta_x(v) \right) dvol(y) \\
&= \int_{T_x^1 X} \left(\int_X f(y) \left(e^{(-i\lambda - \rho) b_v(y)} dvol(y) \right) d\theta_x(v) \right) \\
&= \int_{\partial X} \left(\int_X f(y) e^{(-i\lambda - \rho) B_{\xi,x}(y)} dvol(y) \right) d\lambda_x(\xi) \\
&= \int_{\partial X} \tilde{f}^x(\lambda, \xi) d\lambda_x(\xi).
\end{aligned}
$$

Using the relations (10), namely

$$
\tilde{f}^x(\lambda, \xi) = e^{(i\lambda + \rho) B_{\xi,o}(x)} \tilde{f}^o(\lambda, \xi)
$$

and (3), that is

$$
\frac{d\lambda_x}{d\lambda_o}(\xi) = e^{-2\rho B_{\xi,o}(x)},
$$

we get

$$
\begin{aligned}
\widehat{M_x f}(\lambda) &= \int_{\partial X} e^{(i\lambda + \rho) B_{\xi,o}(x)} \tilde{f}^o(\lambda, \xi) e^{-2\rho B_{\xi,o}(x)} d\lambda_o(\xi) \\
&= \int_{\partial X} \tilde{f}^o(\lambda, \xi) e^{(i\lambda - \rho) B_{\xi,o}(x)} d\lambda_o(\xi).
\end{aligned}
$$

Substituting this last expression for $\widehat{M_x f}(\lambda)$ in the equation

$$
f(x) = C_0 \int_0^\infty \widehat{M_x f}(\lambda) |c(\lambda)|^{-2} d\lambda
$$

gives

$$
f(x) = C_0 \int_0^\infty \int_{\partial X} \tilde{f}^o(\lambda, \xi) e^{(i\lambda - \rho) B_{\xi,o}(x)} d\lambda_o(\xi) |c(\lambda)|^{-2} d\lambda
$$

as required. $\diamond$

The Fourier inversion formula leads immediately to a Plancherel theorem:

Theorem 5.8. *Fix a basepoint $o \in X$. For $f, g \in C_c^\infty(X)$, we have*

$$\int_X f(x)\overline{g(x)} dvol(x) = C_0 \int_0^\infty \int_{\partial X} \tilde{f}^o(\lambda,\xi)\overline{\tilde{g}^o(\lambda,\xi)} d\lambda_o(\xi)|c(\lambda)|^{-2} d\lambda$$

where C_0 is the constant appearing in the Fourier inversion formula.

The Fourier transform $f \mapsto \tilde{f}^o$ extends to an isometry of $L^2(X, dvol)$ into $L^2([0,\infty) \times \partial X, C_0|c(\lambda)|^{-2} d\lambda d\lambda_o(\xi))$.

Proof: Applying the Fourier inversion formula to the function g gives

$$\begin{aligned}
&\int_X f(x)\overline{g(x)} dvol(x) \\
&\quad = C_0 \int_X f(x) \left(\int_0^\infty \int_{\partial X} \overline{\tilde{g}^o(\lambda,\xi)} e^{(-i\lambda-\rho)B_{\xi,o}(x)} d\lambda_o(\xi)|c(\lambda)|^{-2} d\lambda \right) dvol(x) \\
&\quad = C_0 \int_0^\infty \int_{\partial X} \left(\int_X f(x) e^{(-i\lambda-\rho)B_{\xi,o}(x)} dvol(x) \right) \overline{\tilde{g}^o(\lambda,\xi)} d\lambda_o(\xi)|c(\lambda)|^{-2} d\lambda \\
&\quad = C_0 \int_0^\infty \int_{\partial X} \tilde{f}^o(\lambda,\xi)\overline{\tilde{g}^o(\lambda,\xi)} d\lambda_o(\xi)|c(\lambda)|^{-2} d\lambda.
\end{aligned}$$

Taking $f = g$ gives that the Fourier transform preserves L^2 norms,

$$||f||_2 = ||\tilde{f}^o||_2$$

for all $f \in C_c^\infty(X)$. It follows from a standard argument that the Fourier transform extends to an isometry of $L^2(X, dvol)$ into $L^2([0,\infty) \times \partial X, C_0|c(\lambda)|^{-2} d\lambda d\lambda_o(\xi))$. $\diamond$

6. An integral formula for the c-function

In this section we prove the following identity which can be viewed as an analogue of a well-known integral formula for Harish-Chandra's **c**-function (formula (18) in [Hel94], pg. 108):

Theorem 6.1. *Let (X, g) be a simply connected harmonic manifold of purely exponential volume growth and c be the c-function of the radial hypergroup of X. Let $\mathrm{Im}\,\lambda < 0$. Then we have*

$$\lim_{r\to\infty} \frac{\phi_\lambda(r)}{e^{(i\lambda-\rho)r}} = c(\lambda) = \int_{\partial X} e^{-2(i\lambda-\rho)(\xi|\eta)_x} d\lambda_x(\eta).$$

for any $x \in X, \xi \in \partial X$, where $(\xi|\eta)_x$ is the Gromov product given in Lemma 2.3.

For the proof of this identity we need some preparations.

For $x, y, z \in X$, the Gromov product $(y|z)_x$ satisfies the following straightforward consequence of the triangle inequality: let γ be a geodesic joining $y, z \in X$. Then for any point w on this geodesic γ we have

$$(y|z)_x \leq d(x, w).$$

This inequality extends to the boundary:

$$(\xi|\eta)_x \leq d(x, w),$$

for all points w on any geodesic connecting $\xi, \eta \in \partial X$ (where $(\xi|\eta)_x$ is the Gromov product as defined in Lemma 2.3).

If the sectional curvatures of X are bounded above by -1 so that X is a CAT(-1) space, then for $x \in X$ there is a well-known metric on ∂X called the *visual metric* or *Bourdon metric*, defined by $\rho_x(\xi, \eta) = e^{-(\xi|\eta)_x}$ ([Bou96]). In our setting where X is only Gromov hyperbolic, this formula may not define a metric on the boundary, but we can still use the Gromov product to define "balls" in the boundary ∂X with center $\xi \in \partial X$ and radius $r > 0$ by putting

$$B^{(x)}(\xi, r) := \{\eta \in \partial X \mid e^{-(\xi|\eta)_x} < r\}.$$

We need the following geometric result.

Lemma 6.2. *Let $x \in X, \xi \in \partial X$ and let $\gamma_{x,\xi} : [0, \infty) \to X$ be a geodesic ray with $\gamma_{x,\xi}(0) = x$ and $\gamma_{x,\xi}(\infty) = \xi$. Then we have for all $\epsilon \in (0, 1)$, $y = \gamma_{x,\xi}(\log(1/\epsilon))$ and all $\eta \in B^{(x)}(\xi, \epsilon)$:*

$$|B_{\eta,y}(x) - d(x, y)| \le 6\delta. \tag{14}$$

Proof: Let $\eta \in B^{(x)}(\xi, \epsilon)$ be fixed and $R = \log(1/\epsilon)$. Then $(\xi|\eta)_x \ge R$. Let $\gamma_{\xi,\eta} : \mathbb{R} \to X$ be a geodesic connecting ξ and η and $\gamma_{x,\eta} : [0, \infty) \to X$ be a geodesic ray connecting x and η. Let $y_0 = \gamma_{x,\xi}(R - 2\delta)$. Then y_0 is not contained in the δ-tube around $\gamma_{\xi,\eta}(\mathbb{R})$ since $d(x, y_0) = R - 2\delta$ and $d(x, \gamma_{\xi,\eta}(\mathbb{R})) \ge (\xi|\eta)_x \ge R$. Since triangles are δ-thin, y_0 is contained in the δ-tube around $\gamma_{x,\eta}(0, \infty)$. Let $z_0 \in \gamma_{x,\eta}(0, \infty)$ with $d(y_0, z_0) \le \delta$ and, therefore, $d(y, z_0) \le 3\delta$. This implies for $z = \gamma_{x,\eta}(t)$ and $t > 0$ large:

$$\begin{aligned} &|d(x, z) - d(y, z) - d(x, y)| \le \\ &\qquad |d(x, z) - d(z_0, z) - d(x, z_0)| + |d(z_0, z) - d(y, z)| + |d(x, z_0) - d(x, y)| \le 6\delta \end{aligned}$$

since x, z_0, z lie on the geodesic $\gamma_{x,\eta}$ and, therefore, $d(x, z) - d(z_0, z) - d(x, z_0) = 0$ and $|d(z_0, z) - d(y, z)|, |d(x, z_0) - d(x, y)| \le d(y, z_0) \le 3\delta$. The result follows then by taking the limit $t \to \infty$. $\diamond$

This result has the following consequence:

Lemma 6.3. *Let (X, g) be a non-compact simply connected δ-hyperbolic harmonic manifold with horospheres of mean curvature $h > 0$. Then we have for all $x \in X$, $\xi \in \partial X$ and $\epsilon \in (0, 1)$:*

$$\lambda_x(B^{(x)}(\xi, \epsilon)) \le e^{6\delta h} \epsilon^h.$$

Proof: Recall that Gromov hyperbolicity and purely exponential volume growth are equivalent in the setting of non-compact simply connected harmonic manifolds ([Kni12]). We use [KP16, Theorem 1.4] (see also (3)) about the Radon-Nykodym derivative and (14) to obtain for $y = \gamma_{x,\xi}(\log(1/\epsilon))$ with $\gamma_{x,\xi}$ a geodesic ray connecting x and ξ:

$$\begin{aligned} \lambda_x(B^{(x)}(\xi, \epsilon)) &= \int_{B^{(x)}(\xi,\epsilon)} d\lambda_x(\eta) = \int_{B^{(x)}(\xi,\epsilon)} \frac{d\lambda_x}{d\lambda_y} d\lambda_y(\eta) \\ &= \int_{B^{(x)}(\xi,\epsilon)} e^{-hB_{\eta,y}(x)} d\lambda_y(\eta) = \int_{B^{(x)}(\xi,\epsilon)} e^{-hB_{\eta,y}(x)} d\lambda_y(\eta) \\ = \int_{B^{(x)}(\xi,\epsilon)} e^{-hd(x,y)} e^{-h(B_{\eta,y}(x) - d(x,y))} d\lambda_y(\eta) &\le \epsilon^h \int_{B^{(x)}(\xi,\epsilon)} e^{6\delta h} d\lambda_y(\eta) = e^{6\delta h} \epsilon^h. \end{aligned}$$

$\diamond$

With these results we can now present the proof of Theorem 6.1:

Proof: For $\operatorname{Im}\lambda < 0$, using $\phi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda}$ and

$$\Phi_{\pm\lambda}(r) = e^{(\pm i\lambda-\rho)r}(1+o(1)) \quad \text{as } r \to \infty,$$

we have

$$\begin{aligned} \frac{\phi_\lambda(r)}{e^{(i\lambda-\rho)r}} &= c(\lambda)(1+o(1)) + c(-\lambda)e^{-2i\lambda r}(1+o(1)) \\ &\to c(\lambda) \end{aligned} \tag{15}$$

as $r \to \infty$. This proves the first equation in the theorem.

For the second equation in the theorem, we first consider the case $\lambda = it$ where $t \leq -\rho$, so that $\mu := i\lambda - \rho \geq 0$. Fix $x \in X$ and $\xi \in \partial X$. For $\eta \in \partial X$, let $\gamma_{x,\eta} : [0,\infty) \to X$ be the geodesic ray satisfying $\gamma_{x,\eta}(0) = x$ and $\gamma_{x,\eta}(\infty) = \eta$. The normalized surface area measure on the geodesic sphere $S(x,r)$ is given by the push-forward of λ_x under the map $\eta \mapsto \gamma_{x,\eta}(r)$, so by Lemma 5.2

$$\frac{\phi_\lambda(r)}{e^{(i\lambda-\rho)r}} = \int_{\partial X} e^{(i\lambda-\rho)(B_{\xi,x}(\gamma_{x,\eta}(r))-r)} d\lambda_x(\eta)$$

We will apply the dominated convergence theorem to evaluate the limit of the above integral as $r \to \infty$. First note that by Lemma 2.4, for any η not equal to ξ,

$$B_{\xi,x}(\gamma_{x,\eta}(r)) - r \to -2(\xi|\eta)_x$$

as $r \to \infty$, so the integrand converges a.e. as $r \to \infty$,

$$e^{(i\lambda-\rho)(B_{\xi,x}(y(\eta,r))-r)} \to e^{-2(i\lambda-\rho)(\xi|\eta)_x}.$$

Now, using $|B_{\xi,x}(\gamma_{x,\eta}(r))| \leq d(x,\gamma_{x,\eta}(r)) = r$ and $\mu \geq 0$ we have

$$e^{\mu(B_{\xi,x}(\gamma_{x,\eta}(r))-r)} \leq 1.$$

So dominated convergence applies and we conclude that

$$\frac{\phi_\lambda(r)}{e^{(i\lambda-\rho)r}} \to \int_{\partial X} e^{-2(i\lambda-\rho)(\xi|\eta)_x} d\lambda_x(\eta)$$

as $r \to \infty$. This shows the equation

$$c(\lambda) = \int_{\partial X} e^{-2(i\lambda-\rho)(\xi|\eta)_x} d\lambda_x(\eta)$$

for $\lambda = it, t \leq -\rho$. Since $c(\lambda)$ is holomorphic for $\operatorname{Im}\lambda < 0$, we need to show that the right hand side is also holomorphic for $\operatorname{Im}\lambda < 0$. Then both expressions must be equal for $\operatorname{Im}\lambda < 0$, finishing the proof of the theorem.

Since $e^{-2(i\lambda-\rho)(\xi|\eta)_x}$ is holomorphic for all $\lambda \in \mathbb{C}$, we need to show that

$$\int_{\partial X} |e^{-2(i\lambda-\rho)(\xi|\eta)_x}| d\lambda_x(\eta) < \infty$$

for $\operatorname{Im}\lambda < 0$. Then the function $\int_{\partial X} e^{-2(i\lambda-\rho)(\xi|\eta)_x} d\lambda_x(\eta)$ will be holomorphic for $\operatorname{Im}\lambda < 0$ by Morera's Theorem. Let $\lambda = \sigma - i\tau$ with $\sigma \in \mathbb{R}$ and $\tau > 0$. Then we have

$$\begin{aligned}\int_{\partial X} |e^{-2(i\lambda-\rho)(\xi|\eta)_x}| d\lambda_x(\eta) &= \int_{\partial X} e^{-2(\tau-\rho)(\xi|\eta)_x} d\lambda_x(\eta) \\ &= \int_0^\infty \lambda_x(\{\eta \in \partial X \mid e^{-2(\tau-\rho)(\xi|\eta)_x} > t\})dt.\end{aligned}$$

If $\tau \geq \rho$ then the set $\{\eta \in \partial X \mid e^{-2(\tau-\rho)(\xi|\eta)_x} > t\}$ is empty for $t > 1$, and so the last integral reduces to an integral over $[0,1]$, which is bounded above by one since λ_x is a probability measure.

Since X is of purely exponential volume growth, it is a δ-hyperbolic space for some $\delta > 0$ ([Kni12]). For $0 < \tau < \rho$ using Lemma 6.3 and the fact that λ_x is a probability measure we obtain with $h = 2\rho$

$$\begin{aligned}\int_0^\infty \lambda_x(\{\eta \mid e^{-2(\tau-\rho)(\xi|\eta)_x} > t\})dt &\leq 1 + \int_1^\infty \lambda_x(B^{(x)}(\xi, (1/t)^{1/(2(\rho-\tau))}))dt \\ &\leq 1 + e^{6\delta h} \int_1^\infty \left(\frac{1}{t}\right)^{\frac{2\rho}{2(\rho-\tau)}} dt \\ &< \infty.\end{aligned}$$

$\diamond$

7. The convolution algebra of radial functions

In this section, we assume (X,g) to be a non-compact simply connected harmonic manifold without any further assumption unless stated otherwise. Fix a basepoint $o \in X$. We define a notion of convolution with radial functions as follows:

For a function f radial around the point o, let $f = u \circ d_o$, where u is a function on $\mathbb{R}$. For $x \in X$, the *x-translate* of f is defined to be the function

$$\tau_x f = u \circ d_x$$

Note that if $f \in L^1(X, dvol)$, then evaluating integrals in geodesic polar coordinates centered at o and x gives

$$||f||_1 = \int_0^\infty |u(r)|A(r)dr = ||\tau_x f||_1$$

Definition 7.1. *For f an L^1 function on X and g an L^1 function on X which is radial around the point o, the convolution of f and g is the function on X defined by*

$$(f * g)(x) = \int_X f(y)(\tau_x g)(y) dvol(y)$$

Note that, if $g = u \circ d_o$, then

$$
\begin{aligned}
||f * g||_1 &\leq \int_X \int_X |f(y)||(\tau_x g)(y)| dvol(y) dvol(x) \\
&= \int_X |f(y)| \left(\int_X |u(d(x,y))| dvol(x) \right) dvol(y) \\
&= \int_X |f(y)| \left(\int_0^\infty |u(r)| A(r) dr \right) dvol(y) \\
&= ||f||_1 ||g||_1 \\
&< +\infty
\end{aligned}
$$

so that the integral defining $(f * g)(x)$ exists for a.e. x, and $f * g \in L^1(X, dvol)$.

Theorem 7.2. *Let (X, g) be a non-compact simply connected harmonic manifold. Let $L^1_o(X, dvol)$ denote the closed subspace of $L^1(X, dvol)$ consisting of those L^1 functions which are radial around the point o. Then for $f, g \in L^1_o(X, dvol)$ we have $f * g \in L^1_o(X, dvol)$, and $L^1_o(X, dvol)$ forms a commutative Banach algebra under convolution.*

Proof: We first consider functions $f, g \in C_c^\infty(X)$ which are radial around o. It was shown in [PS15, Lemma 2.8] that $f * g$ is again radial around o and it follows from [PS15, Remark 1, p.127] that $f * g = g * f$.

Now the inequality $||f * g||_1 \leq ||f||_1 ||g||_1$ implies, by the density of smooth, compactly supported radial functions in the space $L^1_o(X, dvol)$, that for $f, g \in L^1_o(X, dvol)$ we have $f * g = g * f \in L^1_o(X, dvol)$, so $L^1_o(X, dvol)$ forms a commutative Banach algebra under convolution. $\diamond$

Now we derive a basic identity about the Fourier transform of a convolution. We assume here additionally that (X, g) is of purely exponential volume growth to guarantee the existence of the Fourier transform. Note if $f, g \in C_c^\infty(X)$ with $g = u \circ d_o$ radial around o, then $f * g$ is compactly supported. For the Fourier transform of $f * g$ based at o, using the identity $B_{\xi,o}(x) = B_{\xi,o}(y) + B_{\xi,y}(x)$ we have

$$
\begin{aligned}
\widetilde{f * g}^o(\lambda, \xi) &= \int_X \left(\int_X f(y) u(d(x,y)) dvol(y) \right) e^{(-i\lambda - \rho) B_{\xi,o}(x)} dvol(x) \\
&= \int_X f(y) e^{(-i\lambda - \rho) B_{\xi,o}(y)} \left(\int_X u(d(x,y)) e^{(-i\lambda - \rho) B_{\xi,y}(x)} dvol(x) \right) dvol(y) \\
&= \int_X f(y) e^{(-i\lambda - \rho) B_{\xi,o}(y)} \widetilde{u \circ d_y}^y(\lambda, \xi) dvol(y) \\
&= \tilde{f}^o(\lambda, \xi) \hat{u}(\lambda) \\
&= \tilde{f}^o(\lambda, \xi) \hat{g}(\lambda)
\end{aligned}
$$

where we have used the fact that for the function $u \circ d_y$ which is radial around y we have

$$
\widetilde{u \circ d_y}^y(\lambda, \xi) = \hat{u}(\lambda) = \hat{g}(\lambda)
$$

where $\hat{u}$ is the hypergroup Fourier transform of u and $\hat{g}$ is the spherical Fourier transform of the function g which is radial around o.

Finally, we remark that the radial hypergroup of a harmonic manifold (X, g) of purely exponential volume growth can be realized as the convolution algebra of finite radial measures on the manifold: convolution with radial measures can be defined, and the convolution of two radial measures is again a radial measure. This can be proved by approximating finite radial measures by L^1 radial functions and applying the Theorem 7.2. The convolution algebra $L^1_o(X, dvol)$ is then identified with a subalgebra of the hypergroup algebra of finite radial measures under convolution.

8. The Kunze-Stein phenomenon

In this section we assume that (X, g) is a simply connected harmonic manifold of purely exponential volume growth and we prove a version of the Kunze-Stein phenomenon: for $1 \leq p < 2$, convolution with a radial L^p-function defines a bounded operator on $L^2(X)$.

Lemma 8.1. *Let $x \in X$, let $q > 2$, and let $\gamma_q = 1 - \frac{2}{q}$. Then for any $t \in (-\gamma_q \rho, \gamma_q \rho)$, for any $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda = t$ we have*

$$||\phi_{\lambda,x}||_q \leq ||\phi_{it,x}||_q < +\infty$$

Proof: Given $t \in (-\gamma_q \rho, \gamma_q \rho)$, by Theorem 5.6, for λ with $\operatorname{Im} \lambda = t$, we have for any $y \in X$,

$$\begin{aligned}
|\phi_{\lambda,x}(y)| &= \left| \int_{\partial X} e^{(i\lambda - \rho) B_{\xi,x}(y)} d\lambda_x(\xi) \right| \\
&\leq \int_{\partial X} e^{(-t-\rho) B_{\xi,x}(y)} d\lambda_x(\xi) \\
&= \phi_{it,x}(y)
\end{aligned}$$

hence

$$||\phi_{\lambda,x}||_q \leq ||\phi_{it,x}||_q$$

If $t \neq 0$, then since $\phi_{it,x} = \phi_{-it,x}$, we may as well assume that $t > 0$, in which case we have, letting $r = d(x, y)$,

$$\begin{aligned}
\phi_{it,x}(y) &= c(it)\Phi_{it}(r) + c(-it)\Phi_{-it}(r) \\
&= c(it)e^{(-t-\rho)r}(1 + o(1)) + c(-it)e^{(t-\rho)r}(1 + o(1)) \\
&= c(-it)e^{(t-\rho)r}(1 + o(1))
\end{aligned}$$

as $r \to \infty$, so $|\phi_{it,x}(y)| \leq Ce^{(t-\rho)r}$ for $r \geq M$ for some constants $C, M > 0$. We may also assume $A(r) \leq Ce^{2\rho r}$ for $r \geq M$. Then, evaluating integrals in geodesic polar coordinates centered at x, we have

$$\begin{aligned}\int_{d(x,y)\geq M} |\phi_{it,x}(y)|^q dvol(y) &\leq \int_M^\infty (Ce^{(t-\rho)r})^q (Ce^{2\rho r}) dr \\ &< +\infty\end{aligned}$$

since $(t-\rho)q + 2\rho < 0$ for $0 < t < \gamma_q \rho$, thus $||\phi_{it,x}||_q < +\infty$.

For $t = 0$, applying Hölder's inequality we have, for any $\epsilon > 0$,

$$\begin{aligned}\phi_{0,x}(y) &= \int_{\partial X} e^{-\rho B_{\xi,x}(y)} d\lambda_x(\xi) \\ &= \left(\int_{\partial X} e^{-(1+\epsilon)\rho B_{\xi,x}(y)} d\lambda_x(\xi) \right)^{1/(1+\epsilon)} \\ &= \phi_{i\epsilon,x}(y)^{1/(1+\epsilon)}\end{aligned}$$

from which it follows that by choosing ϵ small enough so that $q/(1+\epsilon) > 2$ we have $||\phi_{0,x}||_q < +\infty$. $\diamond$

We remark that while the spherical Fourier transform was originally defined for radial L^1 functions, after fixing a basepoint $x \in X$ it can also be defined for general L^1 functions by the same formula

$$\hat{g}(\lambda) := \int_X g(y)\phi_{\lambda,x}(y) dvol(y) \ , \ \lambda \in \mathbb{R}$$

We then have the following Lemma:

Lemma 8.2. *Let $x \in X$, let $1 \leq p < 2$ and let g be an L^p-function on X. Let $q > 2$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the spherical Fourier transform $\hat{g}$ of g extends to a holomorphic function of λ on the strip $S_q := \{|\operatorname{Im}\lambda| < \gamma_q \rho\}$, and is bounded on any closed sub-strip $\{|\operatorname{Im}\lambda| \leq t\}$ for $0 < t < \gamma_q\rho$. In particular $\hat{g}$ on $\mathbb{R}$ satisfies a bound*

$$||\hat{g}||_\infty \leq C_p ||g||_p$$

for a constant $C_p > 0$.

Proof: Given $0 < t < \gamma_q \rho$, for any $\lambda \in \mathbb{C}$ with $|\operatorname{Im}\lambda| \leq t$, by the previous Lemma $||\phi_{\lambda,x}||_q \leq C$ for some constant C only depending on q and t, so it follows from Holder's inequality that the function

$$\hat{g}(\lambda) := \int_X g(y)\phi_{\lambda,x}(y) dvol(y)$$

is well-defined and bounded for $|\operatorname{Im}\lambda| \leq t$ by a constant $C_{q,t}$ times $||g||_p$. The holomorphicity of the function $\hat{g}$ follows from Morera's theorem, using the holomorphic dependence of $\phi_{\lambda,x}$ on λ. $\diamond$

We can now prove the following version of the Kunze-Stein phenomenon:

Theorem 8.3. *Let (X, g) be a simply connected harmonic manifold of purely exponential volume growth. Let $x \in X$ and let $1 \leq p < 2$. Let $g \in C_c^\infty(X)$ be radial around the point x. Then for any $f \in C_c^\infty(X)$ we have*

$$||f * g||_2 \leq C_p ||g||_p ||f||_2$$

for some constant $C_p > 0$. It follows that for any $g \in L^p(X)$ radial around x, the map $f \in C_c^\infty(X) \mapsto f * g$ extends to a bounded linear operator on $L^2(X)$ with operator norm at most $C_p ||g||_p$.

Proof: Recall that for $f, g \in C_c^\infty(X)$ with g radial around x, the Fourier transform of a convolution satisfies

$$\widetilde{f * g}^x(\lambda, \xi) = \tilde{f}^x(\lambda, \xi)\hat{g}(\lambda)$$

for $\lambda \in \mathbb{R}, \xi \in \partial X$. Applying the Plancherel theorem and Lemma 8.2 above, we have

$$\begin{aligned} ||f * g||_2 &= ||\widetilde{f * g}^x||_2 \\ &= ||\tilde{f}^x \hat{g}||_2 \\ &\leq ||\hat{g}||_\infty ||\tilde{f}^x||_2 \\ &\leq C_p ||g||_p ||f||_2 \end{aligned}$$

The above inequality, valid for C_c^∞-functions, implies by a standard density argument that for any L^p radial function g, the map $f \in C_c^\infty(X) \mapsto f * g$ extends to a bounded linear operator on $L^2(X)$ with norm at most $C_p ||g||_p$. $\diamond$

References

[ACB97] F. Astengo, R. Camporesi, and B. Di Blasio. The Helgason Fourier transform on a class of nonsymmetric harmonic spaces. *Bull. Austral. Math. Soc. 55*, pages 405–424, 1997.

[BCG95] G. Besson, G. Courtois, and S. Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *Geometric and Functional Analysis Vol. 5*, pages 731–799, 1995.

[Bes78] A. L. Besse. Manifolds all of whose geodesics are closed. *Ergebnisse u.i. Grenzgeb. Math., vol. 93, Springer, Berlin*, 1978.

[BFL92] Y. Benoist, P. Foulon, and F. Labourie. Flots d'Anosov à distributions stable et instable differéntiables. *J. Amer. Math. Soc. 5 (1)*, pages 33–74, 1992.

[BH95] W. R. Bloom and H. Heyer. Harmonic analysis of probability measures on hypergroups. *de Gruyter Studies in Mathematics 20 (Walter de Gruyter, Berlin)*, 1995.

[BH99] M. R. Bridson and A. Haefliger. Metric spaces of non-positive curvature. *Grundlehren der mathematischen Wissenschaften, ISSN 0072-7830; 319*, 1999.

[Bou96] M. Bourdon. Sur le birapport au bord des CAT(-1) espaces. *Inst. Hautes Etudes Sci. Publ. Math. No. 83*, pages 95–104, 1996.

[BS07] S. Buyalo and V. Schroeder. *Elements of asymptotic geometry.* EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2007.

[BX95] W. R. Bloom and Z. Xu. The Hardy-Littlewood maximal function for Chebli-Trimeche hypergroups. *Contemp. Math. 183*, pages 45–69, 1995.

[Che74] H. Chebli. Operateurs de translation generalisee et semi-groupes de convolution. *Theorie de potentiel et analyse harmonique, Springer Lecture Notes in Math., 404*, pages 35–59, 1974.

[Che79] H. Chebli. Theoreme de Paley-Wiener associe a un operateur differentiel singulier sur $(0, \infty)$. *J. Math. Pures Appl. (9) 58*, pages 1–19, 1979.

[CR40] E. T. Copson and H. S. Ruse. Harmonic Riemannian spaces. *Proc. Roy. Soc. Edinburgh 60*, pages 117–133, 1940.
[DK18] Cornelia Druţu and Michael Kapovich. *Geometric group theory*, volume 63 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2018.
[DR92] E. Damek and F. Ricci. A class of nonsymmetric harmonic Riemannian spaces. *Bull. Amer. Math. Soc., N.S. 27 (1)*, pages 139–142, 1992.
[EO73] P. Eberlein and B. O'Neill. Visibility manifolds. *Pacific J. Math.*, 46:45–109, 1973.
[FL92] P. Foulon and F. Labourie. Sur les variétés compactes asymptotiquement harmoniques. *Invent. Math. 109 (1)*, pages 97–111, 1992.
[Heb06] J. Heber. On harmonic and asymptotically harmonic homogeneous spaces. *Geom. Funct. Anal. 16 (4)*, pages 869–890, 2006.
[Hel94] S. Helgason. Geometric analysis on symmetric spaces. *Math. Surveys and Monographs 39 (American Mathematical Society, Providence RI)*, 1994.
[Kap80] A. Kaplan. Fundamental solution for a class of hypoelliptic PDE generated by composition of quadratic forms. *Trans. Amer. Math. Soc. 258*, pages 147–153, 1980.
[Kni02] G. Knieper. Hyperbolic dynamics and Riemannian geometry. *Handbook of Dynamical Systems, Vol.1A, Elsevier Science B., eds. B. Hasselblatt and A. Katok*, pages 453–545, 2002.
[Kni12] G. Knieper. New results on noncompact harmonic manifolds. *Comment. Math. Helv. 87*, pages 669–703, 2012.
[Kni16] G. Knieper. A survey on noncompact harmonic and asymptotically harmonic manifolds. In *Geometry, topology, and dynamics in negative curvature*, volume 425 of *London Math. Soc. Lecture Note Ser.*, pages 146–197. Cambridge Univ. Press, Cambridge, 2016.
[Koo84] T. H. Koornwinder. Jacobi functions and analysis on noncompact semisimple Lie groups. *Special functions: Group theoretical aspects and applications, R. A. Askey et al (eds.), Reidel*, pages 1–85, 1984.
[KP13] G. Knieper and N. Peyerimhoff. Noncompact harmonic manifolds. *Oberwolfach Preprints, https://arxiv.org/pdf/1302.3841.pdf*, 2013.
[KP16] G. Knieper and N. Peyerimhoff. Harmonic functions on rank one asymptotically harmonic manifolds. *J. Geom. Anal.*, 26(2):750–781, 2016.
[Lic44] A. Lichnerowicz. Sur les espaces Riemanniens completement harmoniques. *Bull. Soc. Math. France 72*, pages 146–168, 1944.
[Nik05] Y. Nikolayevsky. Two theorems on harmonic manifolds. *Comment. Math. Helv. 80*, pages 29–50, 2005.
[PS15] N. Peyerimhoff and E. Samiou. Integral geometric properties of non-compact harmonic spaces. *J. Geom. Anal.*, 25(1):122–148, 2015.
[RS02] A. Ranjan and H. Shah. Harmonic manifolds with minimal horospheres. *J. Geom. Anal.*, 12(4):683–694, 2002.
[RS03] A. Ranjan and H. Shah. Busemann functions in a harmonic manifold. *Geom. Dedicata*, 101:167–183, 2003.
[RWW61] H. Ruse, A. Walker, and T. Willmore. Harmonic spaces. *Edizioni Cremonese*, 1961.
[Sza90] Z. Szabo. The Lichnerowicz conjecture on harmonic manifolds. *Journal of Differential Geometry, 31*, pages 1–28, 1990.
[Tri81] K. Trimeche. Transformation integrale de Weyl et theoreme de Paley-Wiener associe a un operateur differentiel singulier sur $(0, \infty)$. *J. Math. Pures Appl. 60*, pages 51–98, 1981.
[Tri97a] K. Trimeche. Generalized wavelets and hypergroups. *Gordon and Breach, Amsterdam*, 1997.
[Tri97b] K. Trimeche. Inversion of the Lions transmutation operators using generalized wavelets. *Appl. Comput. Harmon. Anal. 4, no. 1*, pages 97–112, 1997.
[Wal48] A. C. Walker. On Lichnerowicz's conjecture for harmonic 4-spaces. *J. London Math. Soc. 24*, pages 317–329, 1948.
[Wil93] T. J. Willmore. *Riemannian geometry*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.
[Xu94] Z. Xu. Harmonic analysis on Chebli-Trimeche hypergroups. *Ph.D. thesis, Murdoch University, Australia*, 1994.

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